Professor Jennifer Balakrishnan, *jbala@bu.edu*

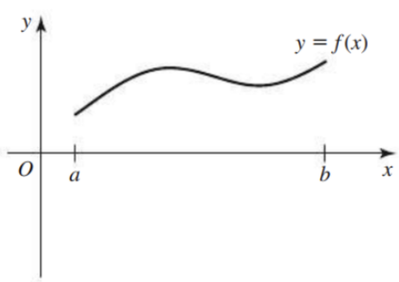
What is on today

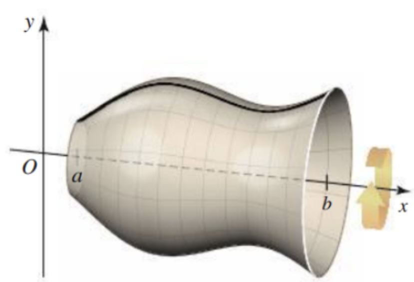
1 Surface area

Briggs-Cochran-Gillett-Schulz §6.6 pp. 457 - 462

In this section, we compute the area of the surface of a solid of revolution. Surface area calculations are important in a number of applications in science and engineering: for instance, in aerodynamics (computing the lift on an airplane wing) and in biology (computing transport rates across cell membranes). Dimension-wise, a surface area problem is between a volume problem (3-dimensional) and an arc length problem (1-dimensional). For this reason, we'll see ideas that appear in both volume and arc length calculations.

Consider a curve $y = f(x)$ on an interval $[a, b]$, where f is a nonnegative function with a continuous first derivative on $[a, b]$. Now revolve the curve about the x-axis to generate a surface of revolution. We would like to find the area of this surface.





Definition 1. *Let f be a nonnegative function with a continuous first derivative on the interval $[a, b]$. The area of the surface generated when the graph of f on the interval $[a, b]$ is revolved about the x-axis is*

$$S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}dx.$$

We may also consider a surface area when a curve is revolved about the y-axis (rather than the x-axis). The result is the same integral, where x is replaced with y, though we must first describe the given curve as a differentiable function of y. If the curve $x = g(y)$ on the interval $[c, d]$ is revolved about the y-axis, the area of the surface generated is

$$S = \int_c^d 2\pi g(y)\sqrt{1 + g'(y)^2}dy.$$

Example 2 (§6.6 Ex. 8, 10, 12)**.** *Find the area of the surface when the given curve is revolved about the given axis.*

1. $y = 12 - 3x$ *for* $1 \leq x \leq 3$*, about the* x*-axis*

2. $y = x^3$ *for* $0 \leq x \leq 1$*, about the* x*-axis*

3. $y = \frac{x^2}{4}$ *for* $2 \leq x \leq 4$*, about the* y*-axis.*

2 Mass and work

Briggs-Cochran-Gillett-Schulz §6.7 pp. 465 - 471

Density is the concentration of mass in an object. An object with *uniform* density satisfies the relationship mass = density · volume. When the density of an object *varies*, we use calculus to compute mass:

Definition 3. *Suppose a thin bar or wire is represented by the interval* $a \leq x \leq b$ *with a* *density function ρ (with units of mass per length). The mass of the object is*

$$m = \int_a^b \rho(x)dx.$$

Example 4 (§6.7 Ex. 14)**.** *Find the mass of the thin bar with density function given by $\rho(x) = 1 + x^3$ for $0 \le x \le 1$.*

Work is the change in energy when a force causes a displacement of an object. If a *constant* force displaces an object a distance in the direction of the force, the work done is the force multiplied by the distance. It's easiest to use metric units for force and work. A newton (N) is the force required to give a 1-kg mass an acceleration of 1 m/s^2. A joule (J) is 1 newton-meter (N-m), the work done by a 1-N force over a distance of 1m.

We use calculus to understand work done with *variable* forces.

Definition 5. *The work done by a variable force F moving an object along a line from $x = a$ to $x = b$ in the direction of the force is*

$$W = \int_a^b F(x)dx.$$

An application of force and work is the stretching and compression of a spring. Suppose an object is attached to a spring on a frictionless horizontal surface. The object slides back and forth under the influence of the spring. We say that the spring is at equilibrium when it is neither compressed nor stretched. It is convenient to let x be the position of the object, where $x = 0$ is the equilibrium position.

According to Hooke's law, the force required to keep the spring in a compressed or stretched position x units from the equilibrium position is $F(x) = kx$, where the positive spring constant k measures the stiffness of the spring. Note that to stretch the spring to a position $x > 0$, a force $F > 0$ (in the positive direction) is required. To compress the spring to a position $x < 0$, a force $F < 0$ (in the negative direction) is required. In other words, the force required to displace the spring is always in the direction of the displacement.

Example 6 (§6.7 Ex. 24)**.** *Suppose a force of 15 N is required to stretch and hold a spring 0.25 m from its equilibrium position.*

1. *Assuming the spring obeys Hooke's law, find the spring constant k.*
2. *How much work is required to compress the spring 0.2 m from its equilibrium position?*
3. *How much additional work is required to stretch the spring 0.3 m if it has already been stretched 0.25 m from its equilibrium position?*

Another type of work problem arises when the motion is vertical and the force is the gravitational force. The gravitational force exerted on an object with mass m is $F = mg$, where $g \approx 9.8$ m/s^2 is the acceleration due to gravity near the surface of Earth. The work in joules required to lift an object of mass m a vertical distance of y meters is

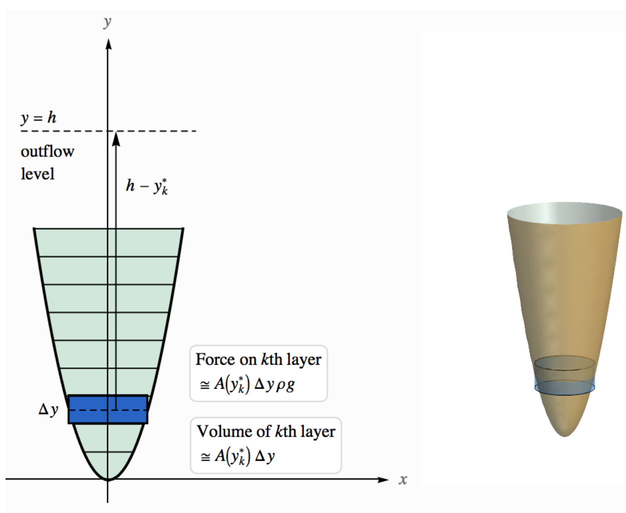
$$\text{work} = \text{Force} \cdot \text{distance} = mgy.$$

We can apply calculus to study lifting problems in the context of rope, a chain, or a body of water. In these situations, different parts of the object are lifted different distances, so we integrate.

Suppose a fluid is pumped out of a tank to a height h above the bottom of the tank. How much work is required, assuming the tank is full of water? Here are three observations:

- Water from different levels of the tank is lifted different vertical distances, requiring different amounts of work.
- Two equal volumes of water from the same horizontal plane are lifted the same distance and require the same amount of work.
- A volume V of water has mass ρV, where $\rho = 1$ g/cm^3 = 1000 kg/m^3 is the density of water.

To solve this problem, we let the y-axis point upward with $y = 0$ at the bottom ("south pole") of the tank. We slice cross sections and compute a Riemann sum:



The cross-sectional area of the kth layer at y_k^*, denoted $A(y_k^*)$, is determined by the shape of the tank. The volume of the kth layer is approximately $A(y_k^*)\Delta y$, so the force on the kth layer is

$$F_k = mg \approx A(y_k^*)\Delta y \cdot \rho \cdot g.$$

To reach the level $y = h$, the kth layer is lifted to an approximate distance $(h - y_k^*)$, so the work in lifting the kth layer to a height h is approximately

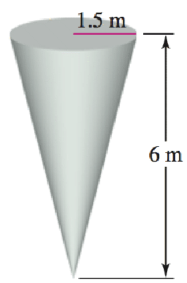
$$W_k = A(y_k^*)\Delta y \rho g(h - y_k^*).$$

Summing all layers and taking the limit as $\Delta y \to 0$ and the number of layers tends to infinity gives us that

$$W = \int_a^b \rho g A(y) D(y) dy$$

computes the work in a lifting problem.

Example 7 (§6.7 Ex. 39)**.** *A water tank is shaped like an inverted cone with height 6m and base radius 1.5 m (see figure).*



1. *If the tank is full, how much work is required to pump the water to the top of the tank and out of the tank?*
2. *Is it true that it takes half as much work to pump the water out of the tank when it is filled to half its depth as when it is full? Explain.*