MA 124 (Calculus II) **Lecture 10: February 25, 2020** **Section A3**

Professor Jennifer Balakrishnan, *jbala@bu.edu*

What is on today

1 Numerical integration

Briggs-Cochran-Gillett-Schulz §8.8 pp. 567 - 577

Sometimes we aren't able to analytically compute definite integrals, and in these situations, we rely on numerical methods. Because numerical methods do not typically produce exact results, we should understand the accuracy of approximations and bound the error.

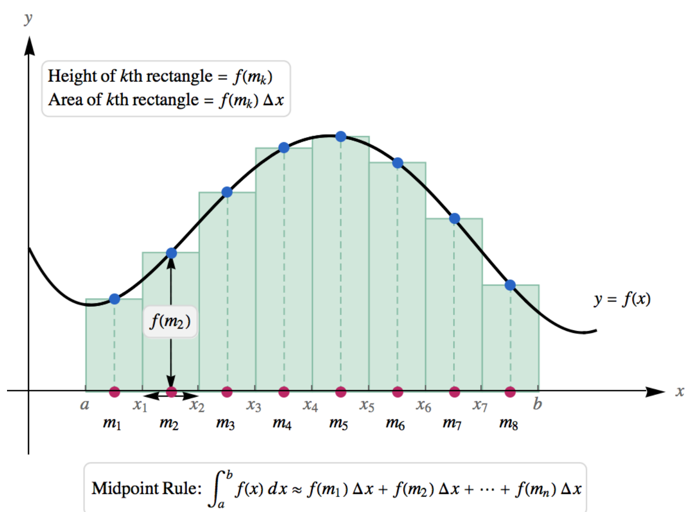
Definition 1. *Suppose c is a computed numerical solution to a problem having an exact solution x. There are two common measures of the error in c as an approximation to x:*

- ***absolute error**: $|c - x|$ and **relative error**: $\frac{|c-x|}{|x|}$*

Example 2 (§8.8 Ex. 12)**.** *Let $x = \sqrt{2}$ and $c = 1.414$. Compute the absolute and relative errors in using c to approximate x.*

Many numerical integration methods are based on the ideas that underlie Riemann sums; these methods approximate the net area of regions bounded by curves.

The Midpoint Rule approximates the region under the curve using rectangles, where the height of the kth rectangle uses the midpoint of the kth subinterval.



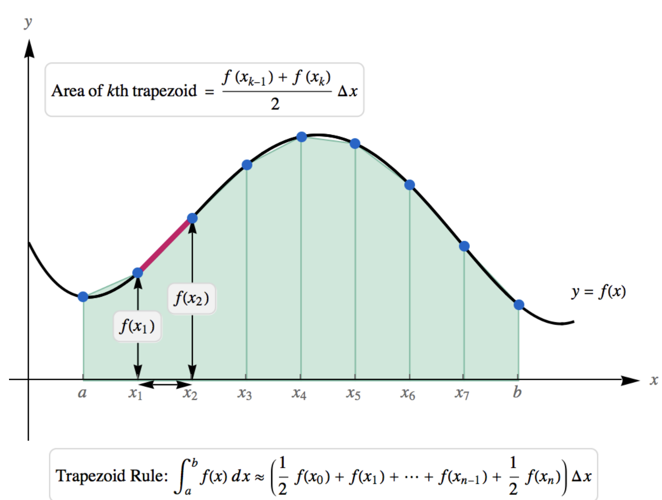
Definition 3. *Suppose f is defined and integrable on $[a, b]$. The Midpoint Rule approximation to $\int_a^b f(x)\,dx$ using n equally spaced subintervals on $[a, b]$ is*

$$M(n) = \sum_{k=1}^{n} f(m_k)\Delta x,$$

where $\Delta x = (b-a)/n, x_k = a + k\Delta x$, and $m_k = (x_{k-1} + x_k)/2$ is the midpoint of $[x_{k-1}, x_k]$ for $k = 1, \ldots, n$.

Example 4 (§8.8 Ex. 16)**.** *Find the Midpoint Rule approximations to $\int_1^9 x^3\,dx$ using $n = 1, 2$, and 4 subintervals.*

The Trapezoid Rule approximates the region under the curve by trapezoids.



The bases of the trapezoids have length Δx. The sides of the kth trapezoid have lengths $f(x_{k-1})$ and $f(x_k)$ for $k = 1, \ldots n$. Therefore, the net area of the kth trapezoid is $\left(\frac{f(x_{k-1})+f(x_k)}{2}\right)\Delta x$.

Definition 5. *Suppose f is defined and integrable on $[a, b]$. The Trapezoid Rule approximation to $\int_a^b f(x)dx$ using n equally spaced subintervals on $[a, b]$ is*

$$T(n) = \left(\frac{1}{2}f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1}{2}f(x_n)\right)\Delta x,$$

where $\Delta x = (b-a)/n$ *and* $x_k = a + k\Delta x$ *for* $k = 0, 1, \ldots, n$.

Note that we have the identity

$$T(2n) = \frac{T(n) + M(n)}{2}.$$

Example 6 (§8.8 Ex. 20)**.** *Find the Trapezoid Rule approximations to* $\int_1^9 x^3\,dx$ *using* $n = 2, 4,$ *and 8 subintervals.*

An improvement over the Midpoint Rule and the Trapezoid Rule results when the graph of f is approximated with curves rather than line segments. Suppose we use three neighboring points on the curve $y = f(x)$, say $(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2))$. These three points determine a parabola, and we can find the net area under the parabola.

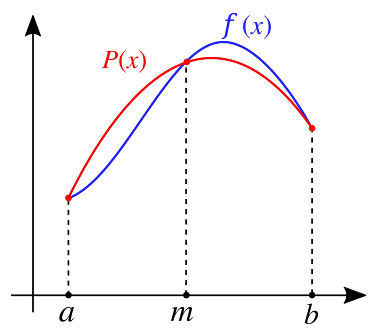


Image credit: Wikipedia, Popletibus, CC BY-SA 4.0

Simpson's Rule computes the net area in this way:

Definition 7. *Suppose* f *is defined and integrable on* $[a, b]$ *and* $n \geq 2$ *is an even integer. The Simpson's Rule approximation to* $\int_a^b f(x)dx$ *using* n *equally spaced subintervals on* $[a, b]$ *is*

$$S(n) = (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n))\frac{\Delta x}{3},$$

where n *is an even integer,* $\Delta x = (b-a)/n$*, and* $x_k = a + k\Delta x$*, for* $k = 0, 1, \ldots, n$.

Note that apart from the first and last terms, the coefficients alternate between 4 and 2; *n must be an even integer* for this rule to apply.

Moreover, note that we have the following relationship among the Midpoint Rule, the Trapezoid Rule, and Simpson's Rule:

$$S(2n) = \frac{2M(n) + T(n)}{3}.$$

Example 8 (§8.8 Ex. 49). *Use Simpson's Rule with $n = 4$ to compute an estimate to*

$$\int_0^4 (3x^5 - 8x^3)\,dx.$$

How good are these numerical approximations? Typically the Midpoint Rule is twice as accurate as the Trapezoid Rule, and Simpson's Rule is more accurate than the former two.

Theorem 9. *Assume that f'' is continuous on the interval $[a, b]$ and that k is a bound on the absolute value of the second derivative of f on $[a, b]$. That is $|f''(x)| \leq k$ for all x in $[a, b]$. Then we have the following:*

$$E_M \leq \frac{k(b-a)}{24}(\Delta x)^2 \qquad E_T \leq \frac{k(b-a)}{12}(\Delta x)^2,$$

where E_M is the error involved in estimating the integral with the Midpoint Rule and E_T is the error involved in estimating the integral with the Trapezoid Rule.

Here is the error estimate for Simpson's Rule:

Theorem 10. *Assume that $f^{(4)}$ is continuous on the interval $[a, b]$ and that K is a bound on the absolute value of the fourth derivative of f on $[a, b]$. That is, $|f^{(4)}(x)| \leq K$ for all x in $[a, b]$. Then we have*

$$E_S \leq \frac{K(b-a)}{180}(\Delta x)^4.$$

Note that the error estimates for the Midpoint Rule and the Trapezoid Rule are second order and Simpson's Rule is fourth order.

2 Improper integrals

Briggs-Cochran-Gillett §8.9 pp. 582 - 586

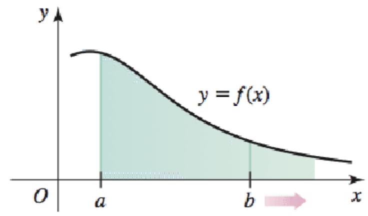
By an *improper integral* we refer to an integral where

- the interval of integration is infinite, or

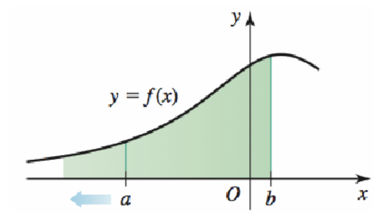
- the integrand is unbounded on the interval of integration.

DEFINITION **Improper Integrals over Infinite Intervals**

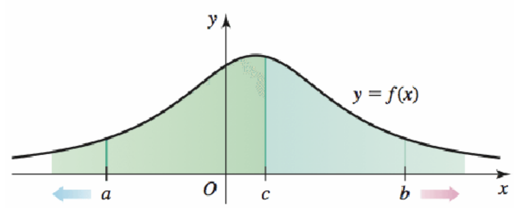
1. If f is continuous on $[a, \infty)$, then $\displaystyle\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$.



2. If f is continuous on $(-\infty, b]$, then $\displaystyle\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$.



3. If f is continuous on $(-\infty, \infty)$, then $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx$ where c is any real number.



If the limits in cases 1–3 exist, the improper integral is said to **converge**; otherwise, they **diverge**.

Example 11 (§8.9 Ex. 7)**.** *Evaluate the integral* $\int_3^{\infty} \frac{dx}{x^2}$ *or state that it diverges.*

Example 12 (§8.9 Ex. 16)**.** *Evaluate the integral* $\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2}$, $a > 0$ *or state that it diverges.*