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What is on today

1 Improper integrals

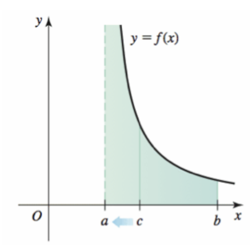
Briggs-Cochran-Gillett-Schulz §8.9 pp. 586 - 590

By an *improper integral* we refer to an integral where

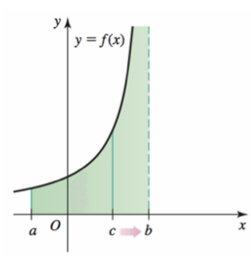
- the interval of integration is infinite, or
- the integrand is unbounded on the interval of integration.

DEFINITION **Improper Integrals with an Unbounded Integrand**

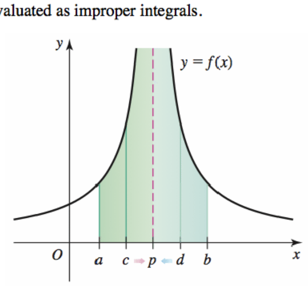
1. Suppose f is continuous on $(a, b]$ with $\lim_{x\to a^+} f(x) = \pm\infty$. Then, $\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$.



2. Suppose f is continuous on $[a, b)$ with $\lim_{x\to b^-} f(x) = \pm\infty$. Then, $\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$.



3. Suppose f is continuous on $[a, b]$ except at the interior point p where f is unbounded. Then, $\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$ where the integrals on the right side are evaluated as improper integrals.



If the limits in cases 1-3 exists, the improper integrals **converge**; otherwise they **diverge**.

Example 1 (§8.9 Ex. 18). *Evaluate the integral* $\int_2^\infty \frac{dx}{(x+2)^2}$ *or state that it diverges.*

Example 2 (§8.9 Ex. 38). *Evaluate the integral* $\int_1^2 \frac{dx}{\sqrt{x-1}}$ *or state that it diverges.*

Example 3 (§8.9 Ex. 66). *Find the volume (if possible) of the solid of revolution given by rotating the region bounded by* $f(x) = (x^2+1)^{-1/2}$ *and the* x*-axis on the interval* $[2, \infty)$ *about the* x*-axis.*

2 Differential equations: direction fields & Euler's method

Briggs-Cochran-Gillett-Schulz §9.2 pp. 606 - 611

Differential equations are a powerful tool used in engineering, the natural and biological sciences, economics, management and finance. A differential equation is an equation involving a function and its derivative: for example, $y'(t) = 3t^2 - 4t + 10$. The goal is to find solutions of the equation: functions y that satisfy the equation.

Now we introduce some terminology for working with differential equations. The **order** of a differential equation is the highest order appearing on a derivative in the equation. For example, $y'' + 16y = 0$ is second order.

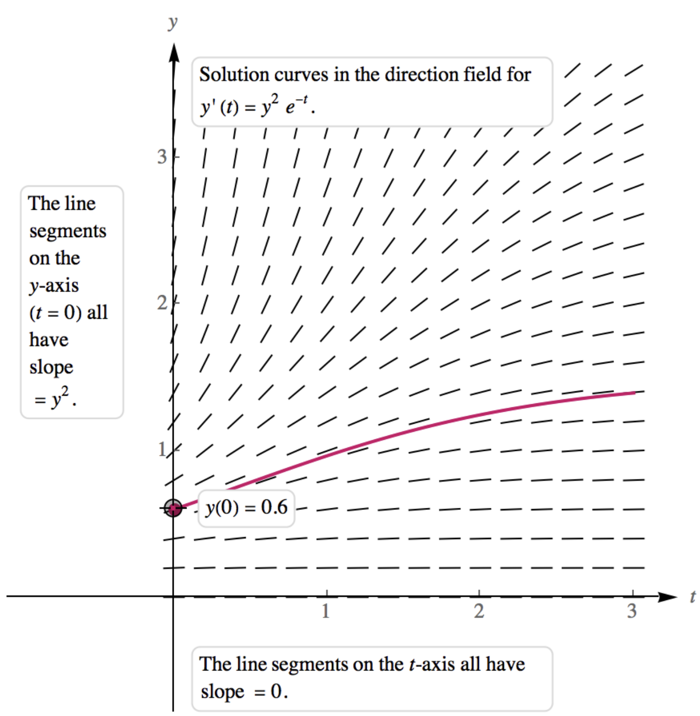
A **first-order linear differential equation** has the form

$$y'(x) + p(x)y(x) = f(x)$$

where p, q, f are given functions that depend only on the independent variable x. A differential equation is often accompanied by initial conditions that specify the values of y, and possibly its derivatives, at a particular point. In general, an nth order equation requires n initial conditions.

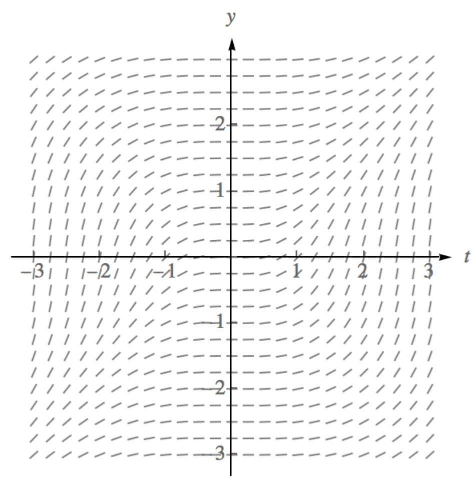
We can understand first-order differential equations geometrically using direction fields. If we consider the first-order differential equation $y'(t) = F(t, y)$, where F is a given expression involving t and or y, a solution of this equation has the property that at each point (t, y) of the solution curve, the slope of the curve is $F(t, y)$. A direction field is a picture that shows the slope of the solution at selected points of the ty-plane.

For example, let's look at the differential equation $y'(t) = y^2e^{-t}$. At each point (t, y), we make a small line segment with slope y^2e^{-t}. The line segment at a point P gives the slope of the solution curve that passes through P, as seen in the figure below. For example, along the t-axis ($y = 0$) the slopes of the line segments are $F(t, 0) = 0$. And along the y-axis ($t = 0$), the slopes of the line segments are $F(0, y) = y^2$.



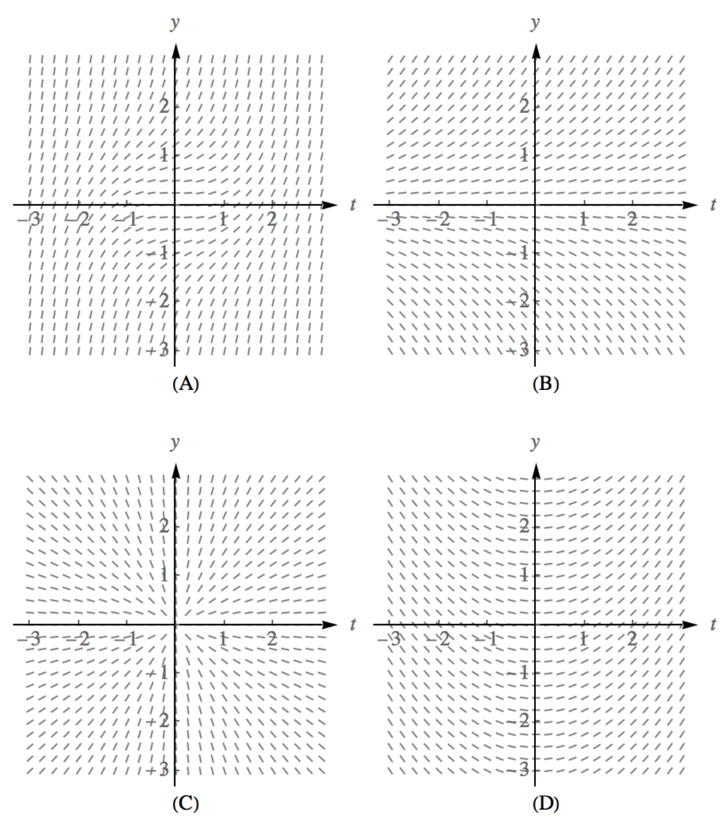
Now suppose an initial condition $y(a) = A$ is given. Above, you can see the curve given by the initial condition of $y(0) = 0.6$. This picks out one particular solution curve. A different initial condition gives a different solution curve.

Example 4 (§9.2 Ex. 7). *Consider $y'(t) = \frac{t^2}{y^2+1}$ with the initial condition of $y(0) = -2$. Sketch a graph of the solution given the direction field below. Repeat this for the initial condition of $y(-2) = 0$.*



Example 5 (§9.2 Ex. 5). *Match equations a-d with the direction fields A-D.*

a. $y'(t) = t/2$
b. $y'(t) = y/2$
c. $y'(t) = (t^2 + y^2)/2$
d. $y'(t) = y/t$



Here is how to sketch a direction field of $y'(t) = f(y)$ by hand:

1. Find the values of y for which $f(y) = 0$. For example, suppose $f(a) = 0$. Then we have $y'(t) = 0$ whenever $y = a$, and the direction field at all points (t, a) consists of horizontal line segments. If the initial condition is $y(0) = a$, then the solution is $y(t) = a$, for all $t \geq 0$. Such a constant solution is called an **equilibrium solution**.
2. Find the values of y for which $f(y) > 0$. For example, suppose $f(b) > 0$. Then $y'(t) > 0$ whenever $y > b$. It follows that the direction field at all points (t, b) has line segments with positive slopes, and the solution is increasing at those points.
3. Find the values of y for which $f(y) < 0$. For example, suppose $f(c) < 0$. Then $y'(t) < 0$ whenever $y < c$. It follows that the direction field at all points (t, c) has line segments with negative slopes, and the solution is decreasing at those points.

As we saw in the the last few examples, a direction field gives us valuable qualitative information about the solutions of a differential equation without solving the equation. Direction fields are also the basis for many computer-based methods for approximating solutions of a differential equation. One begins with the initial condition and advances the solution in small steps, always following the direction field at each time step. The simplest method that uses this idea is called Euler's method.

Euler's method for $y'(t) = f(t, y), y(0) = A$ on $[0, T]$

1. Choose either a time step Δt or a positive integer N such that $\Delta t = \frac{T}{N}$ and $t_k = k\Delta t$ for $k = 0, 1, 2, \ldots, N - 1$.
2. Let $u_0 = y(0) = A$.
3. For $k = 0, 1, 2, \ldots, N - 1$, compute $u_{k+1} = u_k + f(t_k, u_k)\Delta t$. Each u_k is an approximation to the exact solution $y(t_k)$.

Example 6 (§9.2 Ex. 26)**.** *Let $y'(t) = -y, y(0) = -1, \Delta t = 0.2$. Compute the first two approximations u_1 and u_2 given by Euler's method using the given time step.*