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What is on today

1 Sequences and infinite series: an introduction

Briggs-Cochran-Gillett-Schulz §10.1 pp. 639 - 647

A **sequence** $\{a_n\}$ is an ordered list of numbers of the form

$$\{a_1, a_2, a_3, \ldots, \}.$$

Each number in the sequence is called a term of the sequence. A sequence may be generated by a recurrence relation of the form $a_{n+1} = f(a_n)$ for $n = 1, 2, 3, \ldots$, where a_1 is given. A sequence may also be defined with an explicit formula of the form $a_n = f(n)$, for $n = 1, 2, 3, \ldots$.

Example 1 (§10.1 Ex. 24)**.** *Write the first four terms of the sequence $\{a_n\}$ defined by the recurrence relation $a_{n+1} = a_n^2 - 1; a_1 = 1$.*

Perhaps the most important question about a sequence is this: if you go father and farther out in the sequence $a_{100}, \ldots, a_{100000}, \ldots, a_{10000000000}, \ldots$, how do the terms of the sequence behave? Is there a limiting value, or do they grow without bound?

Definition 2. *If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases – that is, if a_n can be made arbitrarily close to L by taking n sufficiently large – then we say $\lim_{n\to\infty} a_n = L$ exists, and the sequence converges to L. If the terms of the sequence do* *not approach a single number as n increases, the sequence has no limit, and the sequence diverges.*

Given a sequence $\{a_1, a_2, a_3, \ldots\}$, the sum of its terms

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$$

is called an infinite series. The sequence of partial sums $\{S_n\}$ associated with this series has the terms

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
S_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k, \quad \text{for } n = 1, 2, 3, \ldots
\end{aligned}$$

If the sequence of partial sums $\{S_n\}$ has a limit L, the infinite series converges to that limit, and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L.$$

If the sequence of partial sums diverges, the infinite series also diverges.

Example 3 (§10.1 Ex. 63)**.** *For the infinite series* $4 + 0.9 + 0.09 + 0.009 + \cdots$*, find the first four terms of the sequence of partial sums. Then make a conjecture about the value of the infinite series.*

Example 4 (§10.1 Ex. 68). *Consider the infinite series* $\sum_{k=1}^{\infty} 2^{-k}$.

1. *Write out the first four terms of the sequence of partial sums.*
2. *Find a formula for the nth partial sum* S_n *of the infinite series. Use this formula to find the next four partial sums* S_5, S_6, S_7, S_8.
3. *Make a conjecture for the values of the series (the limit of* $\{S_n\}$*) or state that it does not exist.*

2 Sequences

Briggs-Cochran-Gillett-Schulz §10.2 pp. 650 - 657

A fundamental question about sequences concerns the behavior of the terms as we go out farther and farther in the sequence. Below we state a few theorems regarding limits of sequences:

Theorem 5 (Limits of sequences from limits of functions). *Suppose* f *is a function such that* $f(n) = a_n$ *for all positive integers* n. *If* $\lim_{x\to\infty} f(x) = L$, *then the limit of the sequence* $\{a_n\}$ *is also* L.

Theorem 6 (Limit laws for sequences). *Assume that the sequences* $\{a_n\}$ *and* $\{b_n\}$ *have limits* A *and* B, *respectively. Then*

1. $\lim_{n\to\infty}(a_n \pm b_n) = A \pm B$
2. $\lim_{n\to\infty} ca_n = cA$, *where* c *is a real number*
3. $\lim_{n\to\infty} a_n b_n = AB$

4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$, *provided* $B \neq 0$

Example 7 (§10.2 Ex. 14)**.** *Find the limit of the sequence* $\{\frac{n^{12}}{3n^{12}+4}\}$ *or determine that the limit does not exist.*

Example 8 (§10.2 Ex. 20)**.** *Find the limit of the sequence* $\{\ln(n^3+1) - \ln(3n^3+10n)\}$ *or determine that the limit does not exist.*

We now introduce some terminology for sequences:

- $\{a_n\}$ is increasing if $a_{n+1} > a_n$; for example, $\{0, 1, 2, 3, \ldots\}$,
- $\{a_n\}$ is nondecreasing if $a_{n+1} \geq a_n$; for example, $\{1, 1, 2, 2, 3, 3, \ldots\}$.
- $\{a_n\}$ is decreasing if $a_{n+1} < a_n$; for example, $\{2, 1, 0, -1, \ldots\}$.
- $\{a_n\}$ is nonincreasing if $a_{n+1} \leq a_n$; for example, $\{0, -1, -1, -2, -2, -3, -3, \ldots\}$.
- $\{a_n\}$ is monotonic if it is either nonincreasing or nondecreasing (it moves in one direction).
- $\{a_n\}$ is bounded if there is a number M such that $|a_n| \leq M$, for all relevant values of n

Geometric sequences have the property that each term is obtained by multiplying the previous term by a fixed constant, called the ratio. They have the form $\{ar^n\}$, where the ratio r and $a \neq 0$ are real numbers.

Theorem 9. *Let r be a real number. Then*

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{if } r \leq -1 \text{ or } r > 1. \end{cases}$$

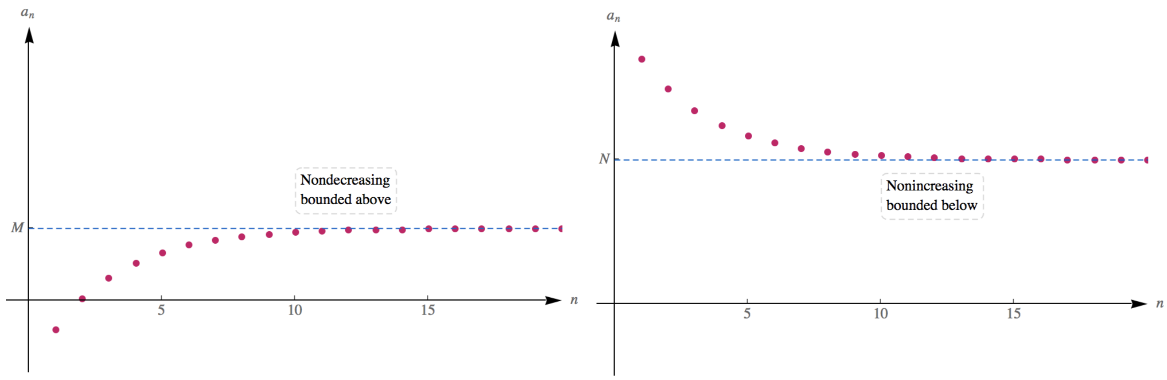
If $r > 0$, then $\{r^n\}$ is a monotonic sequence. If $r < 0$, then $\{r^n\}$ oscillates.

Theorem 10 (Squeeze Theorem for sequences)**.** *Let $\{a_n\}, \{b_n\}$, and $\{c_n\}$ be sequences with $a_n \leq b_n \leq c_n$ for all integers n greater than some index N. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.*

Example 11 (§10.2 Ex. 56)**.** *Find the limit of the sequence $\{\frac{(-1)^n}{n}\}$ or determine that the sequence diverges.*

Example 12 (§10.2 Ex. 65)**.** *Find the limit of the sequence $\{\frac{\cos n}{n}\}$ or determine that the sequence diverges.*

Consider the following bounded monotonic sequences:



Indeed, it is a theorem that all such sequences converge:

Theorem 13. *A bounded monotonic sequence converges.*

We can use earlier results on growth rates of functions (§4.7) to compare growth rates of sequences:

Theorem 14 (Growth Rates of Sequences)**.** *The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, if $\{a_n\}$ appears before $\{b_n\}$ in the list, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$:*

$$\{(\ln n)^q\} \ll \{n^p\} \ll \{n^p(\ln n)^r\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}.$$

The order applies for positive real numbers p, q, r, s and $b > 1$.

Example 15 (§10.2 Ex. 79)**.** *Use the previous theorem to find the limit of the sequence $\left\{\frac{n^{1000}}{2^n}\right\}$ or state that it diverges.*