What is on today

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1 Sequences and infinite series: an introduction

Briggs-Cochran-Gillett-Schulz §10.1 pp. 639 - 647

A sequence \( \{a_n\} \) is an ordered list of numbers of the form

\[
\{a_1, a_2, a_3, \ldots, \}
\]

Each number in the sequence is called a term of the sequence. A sequence may be generated by a recurrence relation of the form \( a_{n+1} = f(a_n) \) for \( n = 1, 2, 3, \ldots \), where \( a_1 \) is given. A sequence may also be defined with an explicit formula of the form \( a_n = f(n) \), for \( n = 1, 2, 3, \ldots \).

Example 1 (§10.1 Ex. 24). Write the first four terms of the sequence \( \{a_n\} \) defined by the recurrence relation \( a_{n+1} = a_n^2 - 1; a_1 = 1 \).

\[
a_1 = 1
a_2 = a_1^2 - 1 = 1^2 - 1 = 0
a_3 = a_2^2 - 1 = 0^2 - 1 = -1
a_4 = a_3^2 - 1 = (-1)^2 - 1 = 0
\]

Perhaps the most important question about a sequence is this: if you go father and farther out in the sequence \( a_{100}, a_{1000}, a_{10000}, \ldots \), how do the terms of the sequence behave? Is there a limiting value, or do they grow without bound?

Definition 2. If the terms of a sequence \( \{a_n\} \) approach a unique number \( L \) as \( n \) increases – that is, if \( a_n \) can be made arbitrarily close to \( L \) by taking \( n \) sufficiently large – then we say \( \lim_{n \to \infty} a_n = L \) exists, and the sequence converges to \( L \). If the terms of the sequence do
not approach a single number as \( n \) increases, the sequence has no limit, and the sequence diverges.

Given a sequence \( \{a_1, a_2, a_3, \ldots\} \), the sum of its terms

\[
a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k
\]

is called an infinite series. The sequence of partial sums \( \{S_n\} \) associated with this series has the terms

\[
S_1 = a_1 \\
S_2 = a_1 + a_2 \\
S_3 = a_1 + a_2 + a_3 \\
\vdots \\
S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k, \quad \text{for } n = 1, 2, 3, \ldots
\]

If the sequence of partial sums \( \{S_n\} \) has a limit \( L \), the infinite series converges to that limit, and we write

\[
\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=1}^{n} a_k = \lim_{n \to \infty} S_n = L.
\]

If the sequence of partial sums diverges, the infinite series also diverges.

**Example 3** (§10.1 Ex. 63). For the infinite series \( 4 + 0.9 + 0.09 + 0.009 + \cdots \), find the first four terms of the sequence of partial sums. Then make a conjecture about the value of the infinite series.

\[
S_1 = 4 \\
S_2 = 4 + 0.9 = 4.9 \\
S_3 = 4 + 0.9 + 0.09 = 4.99 \\
S_4 = 4 + 0.9 + 0.09 + 0.009 = 4.999
\]

Conjecture? We conjecture that the limit of the infinite series is 5.
Example 4 (§10.1 Ex. 68). Consider the infinite series \( \sum_{k=1}^{\infty} 2^{-k} = \frac{1}{2^k} \).

1. Write out the first four terms of the sequence of partial sums.

2. Find a formula for the \( n \)th partial sum \( S_n \) of the infinite series. Use this formula to find the next four partial sums \( S_5, S_6, S_7, S_8 \).

3. Make a conjecture for the values of the series (the limit of \( \{ S_n \} \)) or state that it does not exist.

2 Sequences

Briggs-Cochran-Gillett-Schulz §10.2 pp. 650 - 657

A fundamental question about sequences concerns the behavior of the terms as we go out farther and farther in the sequence. Below we state a few theorems regarding limits of sequences:

Theorem 5 (Limits of sequences from limits of functions). Suppose \( f \) is a function such that \( f(n) = a_n \) for all positive integers \( n \). If \( \lim_{x \to \infty} f(x) = L \), then the limit of the sequence \( \{ a_n \} \) is also \( L \).

Theorem 6 (Limit laws for sequences). Assume that the sequences \( \{ a_n \} \) and \( \{ b_n \} \) have limits \( A \) and \( B \), respectively. Then

1. \( \lim_{n \to \infty} (a_n \pm b_n) = A \pm B \)

2. \( \lim_{n \to \infty} ca_n = cA \), where \( c \) is a real number

3. \( \lim_{n \to \infty} a_nb_n = AB \)
4. \( \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{A}{B} \), provided \( B \neq 0 \)

**Example 7** (§10.2 Ex. 14). Find the limit of the sequence \( \left\{ \frac{n^{12}}{3n^{12}+4} \right\} \) or determine that the limit does not exist.

\[
\lim_{n \to \infty} \frac{n^{12}}{3n^{12}+4} = \lim_{n \to \infty} \frac{\frac{n^{12}}{n^{12}}}{\frac{3n^{12}}{n^{12}} + \frac{4}{n^{12}}} = \lim_{n \to \infty} \frac{1}{3 + \frac{4}{n^{12}}} = \frac{1}{3}
\]

**Example 8** (§10.2 Ex. 20). Find the limit of the sequence \( \left\{ \ln(n^3 + 1) - \ln(3n^3 + 10n) \right\} \) or determine that the limit does not exist.

\[
\ln \left( \frac{n^3 + 1}{3n^3 + 10n} \right) = \ln \left( \frac{n^3 + 1}{3n^3 + 10n} \right) = \ln \left( \frac{n^3 + 1}{3n^3 + 10n} \right) = \ln \left( \frac{\frac{1}{n^3}}{3 + \frac{10}{n^2}} \right) = \ln \left( \frac{1}{3} \right)
\]

\[
\ln \left( \frac{A}{B} \right) = \ln A - \ln B
\]

We now introduce some terminology for sequences:

- \( \{a_n\} \) is increasing if \( a_{n+1} > a_n \); for example, \( \{0, 1, 2, 3, \ldots\} \).
- \( \{a_n\} \) is nondecreasing if \( a_{n+1} \geq a_n \); for example, \( \{1, 1, 2, 2, 3, 3, \ldots\} \).
- \( \{a_n\} \) is decreasing if \( a_{n+1} < a_n \); for example, \( \{2, 1, 0, -1, \ldots\} \).
- \( \{a_n\} \) is nonincreasing if \( a_{n+1} \leq a_n \); for example, \( \{0, -1, -1, -2, -2, -3, -3, \ldots\} \).
- \( \{a_n\} \) is monotonic if it is either nonincreasing or nondecreasing (it moves in one direction).
- \( \{a_n\} \) is bounded if there is a number \( M \) such that \( |a_n| \leq M \), for all relevant values of \( n \).
Geometric sequences have the property that each term is obtained by multiplying the previous term by a fixed constant, called the ratio. They have the form \( \{a r^n\} \), where the ratio \( r \) and \( a \neq 0 \) are real numbers.

**Theorem 9.** Let \( r \) be a real number. Then

\[
\lim_{n \to \infty} r^n = \begin{cases} 
0 & \text{if } |r| < 1 \\
1 & \text{if } r = 1 \\
\text{does not exist} & \text{if } r \leq -1 \text{ or } r > 1.
\end{cases}
\]

If \( r > 0 \), then \( \{r^n\} \) is a monotonic sequence. If \( r < 0 \), then \( \{r^n\} \) oscillates.

**Theorem 10** (Squeeze Theorem for sequences). Let \( \{a_n\} \), \( \{b_n\} \), and \( \{c_n\} \) be sequences with \( a_n \leq b_n \leq c_n \) for all integers \( n \) greater than some index \( N \). If \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L \) then \( \lim_{n \to \infty} b_n = L \).

**Example 11** (§10.2 Ex. 56). Find the limit of the sequence \( \{\left(\frac{-1}{n}\right)^n\} \) or determine that the sequence diverges.

\[
a_1 = \frac{(-1)^1}{1} = -1
\]

\[
a_2 = \frac{(-1)^2}{2} = -\frac{1}{2}
\]

\[
a_3 = \frac{(-1)^3}{3} = \frac{1}{3}
\]

\[
a_4 = \frac{(-1)^4}{4} = \frac{1}{4}
\]

\[
a_5 = \frac{(-1)^5}{5} = -\frac{1}{5}
\]

Example 12 (§10.2 Ex. 65). Find the limit of the sequence \( \{\frac{\cos n}{n}\} \) or determine that the sequence diverges.

\[
a_1 = \frac{\cos 1}{1}
\]

\[
a_2 = \frac{\cos 2}{2}
\]

\[
a_3 = \frac{\cos 3}{3}
\]

\[
\lim_{n \to \infty} \frac{\cos n}{n} = 0
\]
Consider the following bounded monotonic sequences:

Indeed, it is a theorem that all such sequences converge:

**Theorem 13.** A bounded monotonic sequence converges.

We can use earlier results on growth rates of functions (§4.7) to compare growth rates of sequences:

**Theorem 14** (Growth Rates of Sequences). The following sequences are ordered according to increasing growth rates as \( n \to \infty \); that is, if \( \{a_n\} \) appears before \( \{b_n\} \) in the list, then \( \lim_{n \to \infty} \frac{a_n}{b_n} = 0 \) and \( \lim_{n \to \infty} \frac{b_n}{a_n} = \infty \):

\[
\{(\ln n)^q\} \ll \{n^p\} \ll \{n^p(\ln n)^r\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}.
\]

The order applies for positive real numbers \( p, q, r, s \) and \( b > 1 \).

**Example 15** (§10.2 Ex. 79). Use the previous theorem to find the limit of the sequence \( \left\{ \frac{n^{1000}}{2^n} \right\} \) or state that it diverges.

\( n^{1000} : \) this is a polynomial (think: \( x^{1000} \))

\( 2^n : \) this is exponential (think: \( 2^x \))

Exponential has faster growth rate

\[ \lim_{n \to \infty} \frac{n^{1000}}{2^n} = 0 \quad (\text{since } n^{1000} \ll 2^n). \]