Professor Jennifer Balakrishnan, jbala@bu.edu

What is on today

1	Sequences and infinite series: an introduction	1
2	Sequences	9

1 Sequences and infinite series: an introduction

Briggs-Cochran-Gillett-Schulz §10.1 pp. 639 - 647

A sequence $\{a_n\}$ is an ordered list of numbers of the form

 $\{a_1, a_2, a_3, \ldots, \}.$

Each number in the sequence is called a term of the sequence. A sequence may be generated by a recurrence relation of the form $a_{n+1} = f(a_n)$ for $n = 1, 2, 3, \ldots$, where a_1 is given. A sequence may also be defined with an explicit formula of the form $a_n = f(n)$, for $n = 1, 2, 3, \ldots$

Example 1 (§10.1 Ex. 24). Write the first four terms of the sequence $\{a_n\}$ defined by the recurrence relation $a_{n+1} = a_n^2 - 1$; $a_1 = 1$.

$$a_{1} = 1$$

$$a_{2} = a_{1}^{2} - 1 = 1^{2} - 1 = 0$$

$$a_{3} = a_{2}^{2} - 1 = 0^{2} - 1 = -1$$

$$a_{4} = a_{3}^{2} - 1 = (-1)^{2} - 1 = 0$$

Perhaps the most important question about a sequence is this: if you go father and farther out in the sequence $a_{100}, \ldots, a_{100000}, \ldots, a_{1000000000}, \ldots$, how do the terms of the sequence behave? Is there a limiting value, or do they grow without bound?

Definition 2. If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases – that is, if a_n can be made arbitrarily close to L by taking n sufficiently large – then we say $\lim_{n\to\infty} a_n = L$ exists, and the sequence converges to L. If the terms of the sequence do

not approach a single number as n increases, the sequence has no limit, and the sequence

Given a sequence $\{a_1, a_2, a_3, \ldots\}$, the sum of its terms index talls end of sequence $a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k \in K$ for $a_k \in Sequence$ is called an infinite series. The sequence of partial sums $\{S_n\}$ associated with this series has the terms

the terms

$$S_{1} = a_{1}$$

$$S_{2} = a_{1} + a_{2}$$

$$S_{3} = a_{1} + a_{2} + a_{3}$$

$$\vdots$$

$$S_{n} = \underbrace{a_{1} + a_{2} + a_{3} + \dots + a_{n}}_{k=1} = \sum_{k=1}^{n} a_{k}, \text{ for } n = 1, 2, 3, \dots$$

If the sequence of partial sums $\{S_n\}$ has a limit L, the infinite series converges to that limit, , nth partial sum and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=1}^n a_k = \lim_{n \to \infty} S_n = L.$$

If the sequence of partial sums diverges, the infinite series also diverges.

Example 3 (§10.1 Ex. 63). For the infinite series $4 + 0.9 + 0.09 + 0.009 + \cdots$, find the first four terms of the sequence of partial sums. Then make a conjecture about the value of the infinite series.

$$S_1 = 4$$

 $S_2 = 4 \pm 0.9 = 4.9$
 $S_3 = 4 \pm 0.9 \pm 0.09 = 4.99$
 $S_4 = 4 \pm 0.9 \pm 0.09 \pm 0.009 = 4.999$
conjecture? We conjecture that the limit of
the infinite series is 5.

Example 4 (§10.1 Ex. 68). Consider the infinite series $\sum_{k=1}^{\infty} 2^{-k}$.

- 1. Write out the first four terms of the sequence of partial sums.
- 2. Find a formula for the nth partial sum S_n of the infinite series. Use this formula to find the next four partial sums S_5, S_6, S_7, S_8 .
 - 3. Make a conjecture for the values of the series (the limit of $\{S_n\}$) or state that it does not exist.



A fundamental question about sequences concerns the behavior of the terms as we go out farther and farther in the sequence. Below we state a few theorems regarding limits of sequences:

Theorem 5 (Limits of sequences from limits of functions). Suppose f is a function such that $f(n) = a_n$ for all positive integers n. If $\lim_{x\to\infty} f(x) = L$, then the limit of the sequence $\{a_n\}$ is also L.

Theorem 6 (Limit laws for sequences). Assume that the sequences $\{a_n\}$ and $\{b_n\}$ have limits A and B, respectively. Then

- 1. $\lim_{n\to\infty}(a_n\pm b_n)=A\pm B$
- 2. $\lim_{n\to\infty} ca_n = cA$, where c is a real number
- 3. $\lim_{n\to\infty} a_n b_n = AB$

4.
$$\lim_{n \to \infty} \underbrace{\frac{a_n}{b_n}} = \frac{A}{B}$$
, provided $B \neq 0$

Example 7 (§10.2 Ex. 14). Find the limit of the sequence $\{\frac{n^{12}}{3n^{12}+4}\}$ or determine that the limit does not exist.



Example 8 (§10.2 Ex. 20). Find the limit of the sequence $\{\ln(n^3 + 1) - \ln(3n^3 + 10n)\}$ or determine that the limit does not exist.

$$\ln (n^{3} + 1) - \ln (3n^{3} + 10n) = \ln (\frac{n^{3} + 1}{3n^{3} + 10n})$$

$$\lim_{h \to \infty} \ln (\frac{n^{3} + 1}{3n^{3} + 10n}) = \lim_{h \to \infty} \ln (\frac{\frac{n^{3} + 1}{n^{3}}}{\frac{3n^{3} + 10n}{n^{3}}}) = \lim_{h \to \infty} \ln (\frac{1 + \frac{1}{n^{3}}}{3 + \frac{10}{n^{2}}}) = \ln (\frac{1}{3})$$

$$\lim_{h \to \infty} \ln (\frac{1 + \frac{1}{n^{3}}}{\frac{3n^{3} + 10n}{n^{3}}}) = \ln (\frac{1}{3})$$

$$\lim_{h \to \infty} \ln (\frac{1 + \frac{1}{n^{3}}}{\frac{3n^{3} + 10n}{n^{3}}}) = \ln (\frac{1}{3})$$

$$\lim_{h \to \infty} \ln (\frac{1 + \frac{1}{n^{3}}}{\frac{3n^{3} + 10n}{n^{3}}}) = \ln (\frac{1}{3})$$

We now introduce some terminology for sequences:

- $\{a_n\}$ is increasing if $a_{n+1} > a_n$; for example, $\{0, 1, 2, 3, ...\}$,
- $\{a_n\}$ is nondecreasing if $a_{n+1} \ge a_n$; for example, $\{1, 1, 2, 2, 3, 3, ...\}$.
- $\{a_n\}$ is decreasing if $a_{n+1} < a_n$; for example, $\{2, 1, 0, -1, ...\}$.
- $\{a_n\}$ is nonincreasing if $a_{n+1} \le a_n$; for example, $\{0, -1, -1, -2, -2, -3, -3, \ldots\}$.
- $\{a_n\}$ is monotonic if it is either nonincreasing or nondecreasing (it moves in one direction).
- $\{a_n\}$ is bounded if there is a number M such that $|a_n| \leq M$, for all relevant values of n

ratio

a70

Geometric sequences have the property that each term is obtained by multiplying the previous term by a fixed constant, called the ratio. They have the form $\{a_r^n\}$, where the ratio r and $a \neq 0$ are real numbers.

Theorem 9. Let r be a real number. Then

$$\lim_{n \to \infty} r^{n} = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{if } r = 1 \\ \text{if } r \le -1 \text{ or } r > 1. & (-1)^{4} = -1 \\ (-1)^{2} = 4 \\ (-1)^{3} = -1 \\ (-1)^{4} = -1 \\$$

Theorem 10 (Squeeze Theorem for sequences). Let $\{a_n\}, \{b_n\}, and \{c_n\}$ be sequences with $a_n \leq b_n \leq c_n$ for all integers n greater than some index N. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$.

Example 11 (§10.2 Ex. 56). Find the limit of the sequence $\{\frac{(-1)^n}{n}\}$ or determine that the sequence diverges.

there diverges.

$$a_{1} = \frac{(-1)^{1}}{1} = 0$$

$$a_{2} = \frac{(-1)^{2}}{2} = \frac{+1}{2}$$

$$a_{3} = \frac{(-1)^{3}}{5} = 0\frac{1}{3}$$

$$a_{4} = \frac{(-1)^{4}}{4} = \frac{1}{4}$$

$$a_{5} = \frac{(-1)^{5}}{5} = 0\frac{1}{5}$$

$$a_{5} = \frac{(-1)^{5}}{5} = 0\frac{1}{5}$$

$$b_{5} = 0\frac{1}{5}$$

Example 12 (§10.2 Ex. 65). Find the limit of the sequence $\{\frac{\cos n}{n}\}$ or determine that the sequence diverges.

$$a_{1} = \cos t \quad \text{try something else} \cdots$$

$$a_{2} = \cos 2 \quad -1 \leq \cos n \leq 1$$

$$a_{3} = \cos 3 \quad \exists -\frac{1}{h} \leq \frac{\cos n}{h} \leq \frac{1}{h}$$

$$a_{3} = \frac{1}{2} \quad \exists -\frac{1}{h} \leq \frac{\cos n}{h} \leq \frac{1}{h}$$

$$compute \quad \exists \lim_{n \to \infty} \frac{1}{h} = 0 \text{ and } \lim_{n \to \infty} \frac{-1}{h} = 0 \quad \exists \lim_{n \to \infty} \frac{\cos n}{h} = 0.$$

Consider the following bounded monotonic sequences:



Indeed, it is a theorem that all such sequences converge:

Theorem 13. A bounded monotonic sequence converges.

We can use earlier results on growth rates of functions $(\S4.7)$ to compare growth rates of sequences:

Theorem 14 (Growth Rates of Sequences). The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, if $\{a_n\}$ appears before $\{b_n\}$ in the list, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$:

$$\{(\ln n)^q\} \ll \{n^p\} \ll \{n^p(\ln n)^r\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}.$$

The order applies for positive real numbers p, q, r, s and b > 1.

Example 15 (§10.2 Ex. 79). Use the previous theorem to find the limit of the sequence $\left\{\frac{n^{1000}}{2^n}\right\}$ or state that it diverges.

$$n^{1000} : \text{this is a polynomial (think : } x^{1000})$$

$$2^{n} : \text{this is exponential (think : } 2^{\chi})$$

$$exponential has faster growth rate$$

$$\Rightarrow \lim_{n \to \infty} \frac{n^{1000}}{2^{n}} = 0 \quad (since n^{1000} \ll 2^{n}).$$