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What is on today

1 Infinite series

Briggs-Cochran-Gillett-Schulz §10.3 pp. 662 - 667

We will focus our attention on two types of infinite series: geometric series and telescoping series.

Let's recall a few things from last time: given the sequence $\{a_1, a_2, a_3, \ldots\}$, the sum of its terms

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$$

is called an infinite series. The sequence of partial sums $\{S_n\}$ associated with this series has the terms

$$\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
S_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k, \quad \text{for } n = 1, 2, 3, \ldots
\end{aligned}$$

If the sequence of partial sums $\{S_n\}$ has a limit L, the infinite series converges to that limit, and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = L.$$

If the sequence of partial sums diverges, the infinite series also diverges.

As a preliminary step to understanding geometric series, we discuss geometric sums, which are finite sums. A geometric sum with n terms has the form

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k,$$

where $a \neq 0$ and r are real numbers. The number r is called the ratio of the sum. We can compute the value of the geometric sum

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} \tag{1}$$

by doing the following: multiply both sides of (1) by r:

$$rS_n = ar + ar^2 + \cdots + ar^n$$

and consider the difference of the above with (1):

$$S_n - rS_n = a - ar^n.$$

Then solving for S_n gives that

$$S_n = a\frac{1-r^n}{1-r}.$$

Example 1 (§10.3 Ex. 9)**.** *Compute* $\sum_{k=0}^{8} 3^k$.

Example 2 (§10.3 Ex. 14)**.** *Compute* $\sum_{k=0}^{6} \pi^k$.

Now we consider geometric series. The geometric sums $S_n = \sum_{k=0}^{n-1} ar^k$ form the sequence of partial sums for the geometric series $\sum_{k=0}^{\infty} ar^k$. The value of the geometric series is the limit of its sequence of partial sums. Thus we have

$$\sum_{k=0}^{\infty} ar^k = \lim_{n\to\infty} \sum_{k=0}^{n-1} ar^k = \lim_{n\to\infty} a\frac{1-r^n}{1-r}.$$

Then using a theorem from the last class, that

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{if } r \le -1 \text{ or } r > 1, \end{cases}$$

we find that

Theorem 3. *Let $a \neq 0$ and r be real numbers. If $|r| < 1$, then*

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

If $|r| \geq 1$, then the series diverges.

Example 4 (§10.3 Ex. 21, 28, 30, 40)**.** *Evaluate each geometric series or state that it diverges.*

1. $\sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k$.

2. $1 + \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi^3} + \cdots$

3. $\sum_{m=2}^{\infty} \frac{5}{2^m}$.

4. $\sum_{k=1}^{\infty} 3\left(-\frac{1}{8}\right)^{3k}$

Example 5 (§10.3 Ex. 48)**.** *Write the repeating decimal* $0.\overline{09} = 0.09090909\ldots\cdots$ *as a geometric series and then as a rational number.*

We were able to compute geometric series by finding a formula for the sequence of partial sums and then evaluating the limit of the sequence. Not many infinite series can be computed in this way. However, for another class of series, called telescoping series, we can also do something similar. Here are a few examples:

Example 6 (§10.3 Ex. 54, 57)**.** *For the following telescoping series, find a formula for the nth term of the sequence of partial sums S_n. Then evaluate $\lim_{n\to\infty} S_n$ to obtain the value of the series or state that the series diverges.*

1. $\sum_{k=1}^{\infty} \left(\frac{1}{k+2} - \frac{1}{k+3}\right)$

2. $\sum_{k=1}^{\infty} \frac{1}{(k+6)(k+7)}$

We end with several properties of convergent series that will be useful in upcoming work.

Theorem 7 (Properties of convergent series)**.**

1. *Suppose $\sum a_k$ converges to A and c is a real number. The series $\sum ca_k$ converges, and $\sum ca_k = c\sum a_k = cA$.*

2. *Suppose $\sum a_k$ diverges. Then $\sum ca_k$ also diverges, for any real number $c \neq 0$.*

3. *Suppose $\sum a_k$ converges to A and $\sum b_k$ converges to B. The series $\sum(a_k \pm b_k)$ converges, and $\sum(a_k \pm b_k) = \sum a_k \pm \sum b_k = A \pm B$.*

4. *Suppose $\sum a_k$ diverges and $\sum b_k$ converges. Then $\sum(a_k \pm b_k)$ diverges.*

5. *If M is a positive integer, then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=M}^{\infty} a_k$ either both converge or both diverge. In general, whether a series converges does not depend on a finite number of terms added to or removed from the series. However, the value of a convergent series does change if nonzero terms are added or removed.*

Let's practice using these properties:

Example 8 (§10.3 Ex. 77, 82)**.** *Evaluate each series or state that it diverges.*

1. $\sum_{k=1}^{\infty} \frac{(-2)^k}{3^{k+1}}$

2. $\sum_{k=1}^{\infty} \left(2\left(\frac{3}{5}\right)^k + 3\left(\frac{4}{9}\right)^k\right)$.