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What is on today

1 The divergence and integral tests

Briggs-Cochran-Gillett-Schulz §10.4 pp. 671 - 680

As we saw in the last class, for geometric series and telescoping series, the sequence of partial sums can be found and its limit evaluated. But actually, it is difficult or impossible to find an explicit formula for the sequence of partial sums for most infinite series. So it is tough to get the value of most convergent series.

So we try to answer another question: given an infinite series, does it converge? If the answer is no, the series diverges and there are no further questions to ask. But if the answer is yes, the series converges, and it may be possible to estimate its value.

We give a criterion for when an infinite series diverges:

Theorem 1 (Divergence Test)**.** *If* $\sum a_k$ *converges, then* $\lim_{k\to\infty} a_k = 0$*. Equivalently, if* $\lim_{k\to\infty} a_k \neq 0$*, then the series diverges.*

Two key points about this:

1. The Divergence Test CANNOT be used to conclude that a series converges.
2. The Divergence Test does NOT tell us what to conclude if $\lim_{k\to\infty} a_k = 0$.

Example 2 (§10.4 Ex. 9, 10, 14)**.** *Use the Divergence Test to determine whether each of the series below diverges or state that the test is inconclusive.*

1. $\sum_{k=0}^{\infty} \frac{k}{2k+1}$

2. $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$

3. $\sum_{k=1}^{\infty} \frac{k^2}{2^k}$

Our next test comes via a question about the *harmonic series*

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots .$$

Does it converge? If we look at the sequence of partial sums, we have

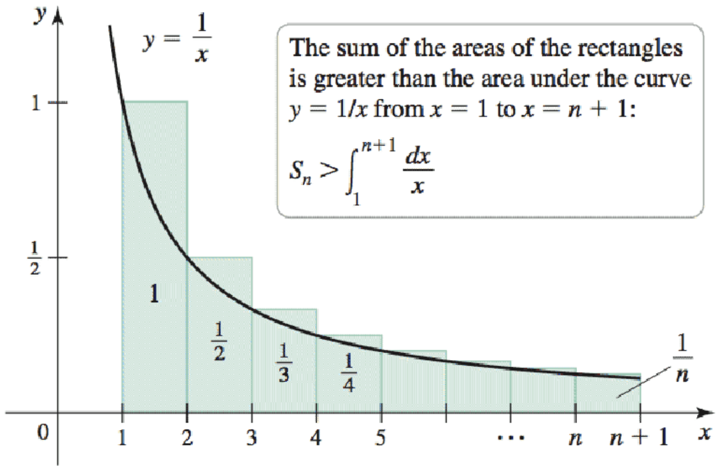
$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 + \frac{1}{2} = \frac{3}{2} \\ S_3 &= 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} \\ S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}, \end{aligned}$$

and there's no obvious pattern in this sequence. In fact, there no simple explicit formula for the S_n.

How do we make sense of the S_n? Observe that

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

is the result of computing a left Riemann sum of the function $y = \frac{1}{x}$ on the interval $[1, n+1]$:



Comparing the sum of the areas of the n rectangles with the area under the curve, we see that

$$S_n > \int_1^{n+1} \frac{dx}{x}.$$

We know that

$$\int_1^{n+1} \frac{dx}{x} = \ln(n+1)$$

increases without bound as n increases. Thus S_n also increases without bound, and the harmonic series diverges.

Theorem 3. *The harmonic series* $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ *diverges, even though the terms of the series approach zero.*

The ideas used to prove that the harmonic series diverges are now used to prove a new convergence test, the Integral Test. This test applies only to series with **positive** terms.

Theorem 4 (Integral Test)**.** *Suppose f is a* ***continuous, positive, decreasing*** *function, for $x \geq 1$, and let $a_k = f(k)$ for $k = 1, 2, 3, \ldots$. Then*

$$\sum_{k=1}^{\infty} a_k \qquad \text{and} \qquad \int_1^{\infty} f(x)dx$$

either both converge or both diverge. In the case of convergence, the value of the integral is not equal to the value of the series.

Example 5 (§10.4 Ex. 20, 21)**.** *Use the Integral Test to determine the convergence or divergence of the following series, after showing that the conditions of the Integral Test are satisfied.*

1. $\sum_{k=1}^{\infty} \frac{e^k}{1+e^{2k}}$

2. $\sum_{k=1}^{\infty} ke^{-2k^2}$

The integral test is used to prove the following:

Theorem 6 (p-series test)**.** *The p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \leq 1$.*

Example 7 (§10.4 Ex. 30, 38)**.** *Determine the convergence or divergence of the following series:*

1. $\sum_{k=2}^{\infty} \frac{k^e}{k^\pi}$

2. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{27k^2}}$

Example 8 (§10.4 Ex. 52, 54)**.** *Determine whether the following series converge or diverge:*

1. $\sum_{k=0}^{\infty} \frac{10}{k^2+9}$

2. $\sum_{k=1}^{\infty} \frac{2^k+3^k}{4^k}$

Theorem 9 (Estimating series with positive terms)**.** *Let f be a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Let $S = \sum_{k=1}^{\infty} a_k$ be a convergent series, and let $S_n = \sum_{k=1}^{n} a_k$ be the sum of the first n terms of the series. The remainder $R_n = S - S_n$ satisfies*

$$R_n < \int_n^{\infty} f(x)dx.$$

Furthermore, the exact value of the series is bounded as follows:

$$S_n + \int_{n+1}^{\infty} f(x)dx < \sum_{k=1}^{\infty} a_k < S_n + \int_n^{\infty} f(x)dx.$$

Example 10 (§10.4 Ex. 41)**.** *Consider the convergent series*

$$\sum_{k=1}^{\infty} \frac{1}{k^6}.$$

1. *Find an upper bound for the remainder in terms of* n.

2. *Find how many terms are needed to ensure the remainder is less than* 10^{-3}.

3. *Find lower and upper bounds on the exact value of the series.*

4. *Find an interval in which the value of the series must lie if you approximate it using ten terms of the series.*