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1 The divergence and integral tests

Briggs-Cochran-Gillett-Schulz §10.4 pp. 671 - 680

The following result gives us a way to approximate a series with all positive terms:

Theorem 1 (Estimating series with positive terms). *Let f be a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \dots$. Let $S = \sum_{k=1}^{\infty} a_k$ be a convergent series, and let $S_n = \sum_{k=1}^n a_k$ be the sum of the first n terms of the series. The remainder $R_n = S - S_n$ satisfies*

$$R_n < \int_n^{\infty} f(x)dx.$$

Furthermore, the exact value of the series is bounded as follows:

$$S_n + \int_{n+1}^{\infty} f(x)dx < \sum_{k=1}^{\infty} a_k < S_n + \int_n^{\infty} f(x)dx.$$

Example 2 (§10.4 Ex. 41). *Consider the convergent series*

$$\sum_{k=1}^{\infty} \frac{1}{k^6}.$$

1. Find an upper bound for the remainder in terms of n .

2. Find how many terms are needed to ensure the remainder is less than 10^{-3} .

3. Find lower and upper bounds on the exact value of the series.

4. Find an interval in which the value of the series must lie if you approximate it using ten terms of the series.

2 The comparison and limit comparison tests

Briggs-Cochran-Gillett-Schulz §10.5 pp. 683 - 686

The Comparison Test lets us leverage information about the convergence or divergence of a series by comparing it to the behavior of another series:

Theorem 3 (Comparison Test). *Let $\sum a_k$ and $\sum b_k$ be series with positive terms.*

1. *If $a_k \leq b_k$ and $\sum b_k$ converges, then $\sum a_k$ converges.*

2. *If $b_k \leq a_k$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.*

Example 4 (§10.5 Ex. 9, 30, 50). *Use the Comparison Test to determine whether the following series converge.*

1. $\sum_{k=1}^{\infty} \frac{1}{k^2+4}$

2.
$$\sum_{k=2}^{\infty} \frac{1}{(k \ln k)^2}$$

3.
$$\sum_{k=2}^{\infty} \frac{5 \ln k}{k}$$

The Comparison Test should be tried if there is an obvious comparison series and the necessary inequality is easily established. But sometimes the right inequality is not easy to establish. In this case, it is often easier to use a more refined test called the Limit Comparison Test:

Theorem 5 (Limit Comparison Test). *Let $\sum a_k$ and $\sum b_k$ be series with positive terms and let $\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = L$.*

1. *If $0 < L < \infty$ (that is, L is a finite positive number), then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.*
2. *If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.*
3. *If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.*

Example 6 (§10.5 Ex. 12, 24, 44). Use the *Limit Comparison Test* to determine whether the following series converge.

1. $\sum_{k=1}^{\infty} \frac{0.0001}{k+4}$

2. $\sum_{k=1}^{\infty} \frac{1}{3^k - 2^k}$

3. $\sum_{k=1}^{\infty} \frac{2^k}{e^k - 1}$