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What is on today

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1 The divergence and integral tests

Briggs-Cochran-Gillett-Schulz §10.4 pp. 671 - 680

The following result gives us a way to approximate a series with all positive terms:

Theorem 1 (Estimating series with positive terms). *Let f be a continuous, positive, decreasing function, for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Let $S = \sum_{k=1}^{\infty} a_k$ be a convergent series, and let $S_n = \sum_{k=1}^{n} a_k$ be the sum of the first n terms of the series. The remainder $R_n = S - S_n$ satisfies*

$$\sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k = \sum_{k=n+1}^{\infty} a_k$$

$$R_n < \int_n^{\infty} f(x)dx.$$

Furthermore, the exact value of the series is bounded as follows:

lower bound

$$S_n + \int_{n+1}^{\infty} f(x)dx < \sum_{k=1}^{\infty} a_k < S_n + \int_n^{\infty} f(x)dx.$$

upper bound

Example 2 (§10.4 Ex. 41). *Consider the convergent series*

$$\sum_{k=1}^{\infty} \frac{1}{k^6}.$$

$\frac{1}{b^5 \cdot -5} = \frac{1}{-5b^5} \to 0$ as $b \to \infty$

1. *Find an upper bound for the remainder in terms of n.*

$R_n < \int_n^{\infty} f(x)dx \quad f(x) = \frac{1}{x^6} \Rightarrow R_n < \int_n^{\infty} \frac{1}{x^6} dx$ want to compute the integral

$\int_n^{\infty} x^{-6} dx = \lim_{b\to\infty} \left(\int_n^b x^{-6} dx\right) = \lim_{b\to\infty} \frac{x^{-5}}{-5}\Big|_n^b = \lim_{b\to\infty} \frac{b^{-5}}{-5} + \frac{n^{-5}}{+5} = \frac{1}{5n^5}$

$R_n < \frac{1}{5n^5}$

2. *Find how many terms are needed to ensure the remainder is less than 10^{-3}.*

$R_n < 10^{-3}$ want n such that

$R_n < \frac{1}{5n^5} < 10^{-3} \Rightarrow \frac{1}{5n^5} < \frac{1}{10^3} \Rightarrow 10^3 < 5n^5 \Rightarrow n=3$

3. *Find lower and upper bounds on the exact value of the series.*

lower bound: $S_n + \int_{n+1}^{\infty} f(x)dx = S_n + \frac{1}{5(n+1)^5}$ (looks very similar to computation in 1)

upper $S_n + \int_n^{\infty} f(x)dx = S_n + \frac{1}{5n^5}$

computed this (in part 1)

$= \lim_{b\to\infty} \frac{x^{-5}}{-5}\Big|_{n+1}^{b}$

$= \lim_{b\to\infty} \cancel{\frac{b^{-5}}{-5}} + \frac{(n+1)^{-5}}{5}$

4. *Find an interval in which the value of the series must lie if you approximate it using ten terms of the series.*

let $n=10$ and compute S_{10}, $\int_{11}^{\infty} f(x)dx$, $\int_{10}^{\infty} f(x)dx$

(want to compute lower bound, upper bound that we got in previous problem, in the case $n=10$)

$1.017341512 = 1/1^6 + 1/2^6 + \frac{1}{3^6} + \dots \frac{1}{10^6}$

$L_{10} \approx S_{10} + \frac{1}{5\cdot 11^5} \approx 1.017342754$

$U_{10} \approx S_{10} + \frac{1}{5\cdot 10^5} \approx 1.01734351 2$

$1.017342754 < \sum_{k=1}^{\infty} a_k < 1.017343512$ $\left(= \frac{1}{k^6}\right)$

these are lower and upper bounds for the true value of infinite series.

2 The comparison and limit comparison tests

Briggs-Cochran-Gillett-Schulz §10.5 pp. 683 - 686

The Comparison Test lets us leverage information about the convergence or divergence of a series by comparing it to the behavior of another series:

Theorem 3 (Comparison Test). *Let $\sum a_k$ and $\sum b_k$ be series with positive terms.*

1. *If $a_k \le b_k$ and $\sum b_k$ converges, then $\sum a_k$ converges.*
2. *If $b_k \le a_k$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.*

$\sum a_k \le \sum b_k$ ← converges

Example 4 (§10.5 Ex. 9, 30, 50). *Use the Comparison Test to determine whether the following series converge.*

1. $\sum_{k=1}^{\infty} \frac{1}{k^2+4}$

$\frac{1}{k^2+4}$ compare to $\frac{1}{k^2}$ ⇝ we have $\sum \frac{1}{k^2}$ (converges by p-series series with $p=2$)

$k^2+4 > k^2$

$\Rightarrow \frac{1}{k^2+4} < \frac{1}{k^2}$ and $\sum \frac{1}{k^2}$ converges $\Rightarrow \sum \frac{1}{k^2+4}$ converges by Comp. Test

2. $\sum_{k=2}^{\infty} \frac{1}{(k \ln k)^2}$

what can we compare this to?

$(k \ln k)^2 = k^2 (\ln k)^2 > (\ln k)^2$ to use; have to know $\sum \frac{1}{(\ln k)^2}$??

$> k^2 \ln k \rightsquigarrow \sum \frac{1}{k^2 \ln k}$??

$> k \ln k \rightsquigarrow \sum \frac{1}{k \ln k}$??

$\boxed{> k^2 \text{ for } k \geq 3 \rightsquigarrow \sum \frac{1}{k^2}}$

Since $k^2 (\ln k)^2 > k^2$ for $k \geq 3$

$\Rightarrow \frac{1}{k^2 (\ln k)^2} < \frac{1}{k^2}$

$\Rightarrow \sum \frac{1}{k^2 (\ln k)^2} < \sum \frac{1}{k^2}$ and $\sum \frac{1}{k^2}$ converges by p-series test

$\Rightarrow \sum \frac{1}{(k \ln k)^2}$ converges by Comparison Test.

3. $\sum_{k=2}^{\infty} \frac{5 \ln k}{k}$

What can we compare $\frac{5 \ln k}{k}$ to? $\frac{1}{k} \leftarrow \sum \frac{1}{k}$ diverges

$\frac{5 \ln k}{k} > \frac{1}{k}$ for $k \geq 3 \Rightarrow \sum \frac{5 \ln k}{k} > \sum \frac{1}{k}$ for $k \geq 3$

and this implies by Comp. Test that it diverges.

↳ true if

$5 \ln k > 1$

$\ln k > 1/5$ for $k \geq 3$.

The Comparison Test should be tried if there is an obvious comparison series and the necessary inequality is easily established. But sometimes the right inequality is not easy to establish. In this case, it is often easier to use a more refined test called the Limit Comparison Test:

Theorem 5 (Limit Comparison Test). *Let $\sum a_k$ and $\sum b_k$ be series with positive terms and let $\lim_{k\to\infty} \frac{a_k}{b_k} = L$.*

1. *If $0 < L < \infty$ (that is, L is a finite positive number), then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.* (not zero)
2. *If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.*
3. *If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.*

Example 6 (§10.5 Ex. 12, 24, 44)**.** *Use the Limit Comparison Test to determine whether the following series converge.* use $\sum \frac{1}{k}$ to compare

1. $\sum_{k=1}^{\infty} \frac{0.0001}{k+4}$ ① use $\frac{1}{k}$ to compare

$\lim_{k\to\infty} \frac{a_k}{b_k} =$

$$\lim_{k\to\infty} \frac{\frac{0.0001}{k+4}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{(0.0001k)\cdot\frac{1}{k}}{(k+4)\cdot\frac{1}{k}} = \lim_{k\to\infty} \frac{0.0001}{1+\underbrace{4/k}_{0}} = 0.0001$$

finite nonzero (case 1)

$\sum \frac{1}{k}$ diverges and by the Limit Comparison Test, $\sum \frac{0.0001}{k+4}$ diverges as well.

2. $\sum_{k=1}^{\infty} \frac{1}{3^k-2^k}$ What can we use to compare? $\sum \frac{1}{3^k}$ geometric converges.

$$\lim_{k\to\infty} \frac{\frac{1}{3^k}}{\frac{1}{3^k-2^k}} = \lim_{k\to\infty} \frac{(3^k-2^k)/3^k}{3^k/3^k} = \lim_{k\to\infty} \frac{1^k - \frac{2^k}{3^k}}{1} = \lim_{k\to\infty} 1^k - \underbrace{\left(\frac{2}{3}\right)^k}_{\to 0} = 1$$

and so by Limit Comparison Test, since $\sum \frac{1}{3^k}$ converges, $\sum \frac{1}{3^k-2^k}$ converges as well.

3. $\sum_{k=1}^{\infty} \frac{2^k}{e^k-1}$