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What is on today

1 The comparison and limit comparison tests, wrap up

Briggs-Cochran-Gillett-Schulz §10.5 pp. 683 - 686

Recall our two tests from last class:

Theorem 1 (Comparison Test). *Let $\sum a_k$ and $\sum b_k$ be series with **positive** terms.*

1. *If $a_k \leq b_k$ and $\sum b_k$ converges, then $\sum a_k$ converges.*
2. *If $b_k \leq a_k$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.*

Theorem 2 (Limit Comparison Test). *Let $\sum a_k$ and $\sum b_k$ be series with **positive** terms and let $\lim_{k\to\infty} \frac{a_k}{b_k} = L$.*

1. *If $0 < L < \infty$ (that is, L is a finite positive number), then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.*
2. *If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.*
3. *If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.*

Example 3 (§10.5 Ex. 51, 40). *Use the test of your choice to determine whether the following series converge.*

1. $\sum_{k=1}^{\infty} \frac{k^8}{k^{11}+3}$

2. $\left(\frac{1}{2}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{3}{4}\right)^4 + \cdots$

2 Alternating series

Briggs-Cochran-Gillett-Schulz §10.6 pp. 688 - 694

The previous tests focused on infinite series with positive terms. We shift our attention to studying series with terms that have strictly alternating signs, as in the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots .$$

The factor $(-1)^{k+1}$ (or possibly $(-1)^k$) provides the alternating signs.

Theorem 4 (Alternating Series Test)**.** *The alternating series $\sum(-1)^{k+1}a_k$ converges if*

1. *the terms of the series are nonincreasing in magnitude ($0 < a_{k+1} \leq a_k$, for k greater than some index N) and*
2. $\lim_{k\to\infty} a_k = 0$.

What does the Alternating Series Test tell us about the alternating harmonic series?

Theorem 5. *The alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ converges.*

For series of **positive** terms, $\lim_{k\to\infty} a_k = 0$ does **NOT** imply convergence. For **alternating series with nonincreasing** terms, $\lim_{k\to\infty} a_k = 0$ **DOES** imply convergence.

Example 6 (§10.6 Ex. 16, 20, 24)**.** *Determine whether the following series converge.*

1. $\sum_{k=0}^{\infty} \frac{(-1)^k}{k^2+10}$

2. $\sum_{k=0}^{\infty} \left(-\frac{1}{5}\right)^k$

3. $\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln^2 k}$

Recall that if a series converges to a value S, then the remainder is $R_n = S - S_n$, where S_n is the sum of the first n terms of the series. An upper bound on the magnitude of the remainder (the *absolute error*) in an alternating series arises form the following observation: when the terms are nonincreasing in magnitude, the value of the series is always trapped between successive terms of the sequence of partial sums. Thus we have

$$|R_n| = |S - S_n| \leq |S_{n+1} - S_n| = a_{n+1}.$$

This justifies the following theorem:

Theorem 7 (Remainder in Alternating Series)**.** *Let $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ be a convergent alternating series with terms that are nonincreasing in magnitude. Let $R_n = S - S_n$ be the remainder. Then $|R_n| \leq a_{n+1}$.*

Example 8 (§10.6 Ex. 35)**.** *Determine how many terms of the convergent series $\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$ must be summed to be sure that the remainder is less than 10^{-4} in magnitude. (Although you do not need it, the exact value is $\pi/4$.)*

Now we will consider infinite series $\sum a_k$ where the terms are allowed to be any real numbers (not just all positive or alternating). We first introduce some terminology:

Definition 9. *If $\sum |a_k|$ converges, then we say that $\sum a_k$ converges absolutely. If $\sum |a_k|$ diverges and $\sum a_k$ converges, then $\sum a_k$ converges conditionally.*

The series $\sum \frac{(-1)^{k+1}}{k^2}$ is an example of an absolutely convergent series because the series of absolute values $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is a convergent p-series. On the other hand, the alternating harmonic series $\sum \frac{(-1)^{k+1}}{k}$ is an example of a conditionally convergent series since the series of absolute values $\sum_{k=1}^{\infty}\frac{1}{k}$ is the harmonic series, which diverges.

Theorem 10. *If $\sum |a_k|$ converges, then $\sum a_k$ converges (absolute convergence implies convergence). Equivalently, if $\sum a_k$ diverges, then $\sum |a_k|$ diverges.*

Example 11 (§10.6 Ex. 45, 48, 53)**.** *Determine whether the following series converge absolutely, converge conditionally, or diverge.*

1. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2/3}}$

2. $\sum_{k=1}^{\infty} \left(-\frac{1}{3}\right)^k$

3. $\sum_{k=1}^{\infty} (-1)^k \tan^{-1} k$