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What is on today

1 The comparison and limit comparison tests, wrap up

Briggs-Cochran-Gillett-Schulz §10.5 pp. 683 - 686

Recall our two tests from last class:

Theorem 1 (Comparison Test). *Let $\sum a_k$ and $\sum b_k$ be series with **positive** terms.*

1. *If $a_k \le b_k$ and $\sum b_k$ converges, then $\sum a_k$ converges.*
2. *If $b_k \le a_k$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.*

Theorem 2 (Limit Comparison Test). *Let $\sum a_k$ and $\sum b_k$ be series with **positive** terms and let $\lim_{k\to\infty} \frac{a_k}{b_k} = L$.*

1. *If $0 < L < \infty$ (that is, L is a finite positive number), then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.*
2. *If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.*
3. *If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.*

Example 3 (§10.5 Ex. 51, 40). *Use the test of your choice to determine whether the following series converge.*

1. $\sum_{k=1}^{\infty} \frac{k^8}{k^{11}+3}$

Idea: $\frac{k^8}{k^{11}+3} \approx \frac{k^8}{k^{11}} = \frac{1}{k^3}$. use $\frac{1}{k^3}$ to compare

Limit Comparison Test:

$$\lim_{k\to\infty} \frac{\frac{k^8}{k^{11}+3}}{\frac{1}{k^3}} = \lim_{k\to\infty} \frac{k^{11}}{k^{11}+3} = \lim_{k\to\infty} \frac{1}{1+3/k^{11}} = 1$$

so $\sum \frac{k^8}{k^{11}+3}$ has same behavior as $\sum \frac{1}{k^3}$ (which converges by p-series test w/ $p=3$)

$\Rightarrow$ converges by Limit Comparison Test.

Comparison Test?

$\boxed{\frac{k^8}{k^{11}+3} \overset{?}{<} \frac{1}{k^3}}$ ✓ (want this since $\sum \frac{1}{k^3}$ conv. and so want our series to be smaller)

$\Rightarrow \frac{k^3 \cdot k^8}{k^{11}+3} < 1 \Rightarrow k^3 \cdot k^8 < k^{11}+3$

$k^{11} < k^{11}+3$ ✓

Instead had $\frac{k^8}{k^{11}-3} < \frac{1}{k^3} \Rightarrow \cancel{k^{11} < k^{11}-3}$??

$\Rightarrow$ over w/ different comparison or use L.C.T.

2. $\left(\frac{1}{2}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{3}{4}\right)^4 + \cdots = \sum_{n=2}^{\infty} \left(\frac{n-1}{n}\right)^n = \sum_{n=2}^{\infty} \left(1-\frac{1}{n}\right)^n$ So $\left\{\left(1-\frac{1}{n}\right)^n\right\} \to e^{-1}$ as $n \to \infty$

$\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = L$??

$\log\left(\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n\right) = \log L$

$\|$

$\lim_{n\to\infty} n \cdot \log\left(1-\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\log\left(1-\frac{1}{n}\right)}{\frac{1}{n}} \overset{\text{"}\frac{0}{0}\text{" L'Hopital's Rule}}{=} \frac{\frac{-(-1)n^{-2}}{1-\frac{1}{n}}}{-n^{-2}} = \frac{-1}{1-\frac{1}{n}} = -1. = \log L$

$\Rightarrow \boxed{L = e^{-1}}$

$\boxed{\text{So the infinite series diverges by Divergence Test.}}$

2 Alternating series

Briggs-Cochran-Gillett-Schulz §10.6 pp. 688 - 694

The previous tests focused on infinite series with positive terms. We shift our attention to studying series with terms that have strictly alternating signs, as in the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots .$$

The factor $(-1)^{k+1}$ (or possibly $(-1)^k$) provides the alternating signs.

Theorem 4 (Alternating Series Test)**.** *The alternating series $\sum(-1)^{k+1}a_k$ converges if*

1. *the terms of the series are nonincreasing in magnitude ($0 < a_{k+1} \le a_k$, for k greater than some index N) and*
2. $\lim_{k\to\infty} a_k = 0$.

decreasing : $a_{k+1} < a_k$

non-increasing: $a_{k+1} \le a_k$

What does the Alternating Series Test tell us about the alternating harmonic series?

Theorem 5. *The alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ converges.*

For series of **positive** terms, $\lim_{k\to\infty} a_k = 0$ does **NOT** imply convergence. For **alternating series with nonincreasing** terms, $\lim_{k\to\infty} a_k = 0$ **DOES** imply convergence.

(bec · the harmonic series $\sum \frac{1}{k}$ diverges!)

Example 6 (§10.6 Ex. 16, 20, 24). *Determine whether the following series converge.*

1. $\sum_{k=0}^{\infty} \frac{(-1)^k}{k^2+10}$

are the terms nonincreasing in magnitude?

$\frac{1}{(k+1)^2+10} \leq \frac{1}{k^2+10}$? ✓

$\lim_{k\to\infty} \frac{1}{k^2+10} = 0$ ✓

$\Rightarrow$ by Alternating Series Test, converges.

2. $\sum_{k=0}^{\infty} \left(-\frac{1}{5}\right)^k = (-1)^k \cdot \left(\frac{1}{5}\right)^k$

terms nonincreasing? ✓

limit $\lim_{n\to\infty} \left(\frac{1}{5}\right)^n = 0$ (result from geometric series: if $|r|<1$, then lim is 0)

$\Rightarrow$ by Alternating Series Test, it converges.

3. $\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln^2 k}$

terms nonincreasing? (e.g. $\frac{1}{5\ln^2 5}, \frac{1}{6\ln^2 6}, \frac{1}{7\ln^2 7}, \ldots$)

yes.

$\lim_{k\to\infty} \frac{1}{k\cdot \ln^2 k} = 0$

$\Rightarrow$ by Alternating Series Test, it converges.

Recall that if a series converges to a value S, then the remainder is $R_n = S - S_n$, where S_n is the sum of the first n terms of the series. An upper bound on the magnitude of the remainder (the *absolute error*) in an alternating series arises form the following observation: when the terms are nonincreasing in magnitude, the value of the series is always trapped between successive terms of the sequence of partial sums. Thus we have

$$|R_n| = |S - S_n| \leq |S_{n+1} - S_n| = a_{n+1}.$$

$\boxed{S_n = \sum_{k=1}^{n} a_k \cdot (-1)^{k+1}}$

This justifies the following theorem:

Theorem 7 (Remainder in Alternating Series)**.** *Let $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ be a convergent alternating series with terms that are nonincreasing in magnitude. Let $R_n = S - S_n$ be the remainder. Then $|R_n| \leq a_{n+1}$.*

Example 8 (§10.6 Ex. 35)**.** *Determine how many terms of the convergent series $\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$ must be summed to be sure that the remainder is less than 10^{-4} in magnitude. (Although you do not need it, the exact value is $\pi/4$.)*

$S_n = \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1}$

Theorem $\boxed{|R_n| \leq a_{n+1} = \frac{1}{2 \cdot n+1} < 10^{-4}}$

next term in series

(we computed (n+1)st term

$\Rightarrow 10^4 < 2n+1 \Rightarrow \frac{10^4-1}{2} < n$

first unused term $a_{n+1} = \frac{1}{2n+1}$ is using term with $k=n$

$\frac{10\,000-1}{2}$

$n > \frac{9999}{2} = 4999.5 \Rightarrow n = 5000$

Now we will consider infinite series $\sum a_k$ where the terms are allowed to be any real numbers (not just all positive or alternating). We first introduce some terminology:

Definition 9. *If $\sum |a_k|$ converges, then we say that $\sum a_k$ converges absolutely. If $\sum |a_k|$ diverges and $\sum a_k$ converges, then $\sum a_k$ converges conditionally.*

The series $\sum \frac{(-1)^{k+1}}{k^2}$ is an example of an absolutely convergent series because the series of absolute values $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is a convergent p-series. On the other hand, the alternating harmonic series $\sum \frac{(-1)^{k+1}}{k}$ is an example of a conditionally convergent series since the series of absolute values $\sum_{k=1}^{\infty}\frac{1}{k}$ is the harmonic series, which diverges.

Theorem 10. *If $\sum |a_k|$ converges, then $\sum a_k$ converges (absolute convergence implies convergence). Equivalently, if $\sum a_k$ diverges, then $\sum |a_k|$ diverges.*

Example 11 (§10.6 Ex. 45, 48, 53). *Determine whether the following series converge absolutely, converge conditionally, or diverge.*

1. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2/3}}$

does $\sum \frac{1}{k^{2/3}}$ converge? no. p-series test w/ $p = 2/3$. diverges.

does $\sum \frac{1}{k^{2/3}}(-1)^k$ converge? yes by A.S.T.

↳ are the terms $\frac{1}{k^{2/3}}$ non-increasing? ✓

what is $\lim_{k\to\infty} \frac{1}{k^{2/3}} = 0$ $\Rightarrow$ converges

Converges conditionally.

2. $\sum_{k=1}^{\infty} \left(-\frac{1}{3}\right)^k = (-1)^k \cdot \left(\frac{1}{3}\right)^k$ geometric

does $\sum \left(\frac{1}{3}\right)^k$ converge? yes, geometric!

does $\sum (-1)^k \left(\frac{1}{3}\right)^k$ converge?

absolute convergence $\Rightarrow$ convergence

$\boxed{\text{converges absolutely}}$.

3. $\sum_{k=1}^{\infty} (-1)^k \tan^{-1} k$

does $\sum_{k=1}^{\infty} \tan^{-1} k$ converge?

$\lim_{k\to\infty} \tan^{-1} k = \frac{\pi}{2}$

so series diverges. by divergence test.

does $\sum_{k=1}^{\infty} (-1)^k \tan^{-1} k$ converge?

$\lim_{k\to\infty} (-1)^k \tan^{-1} k \neq 0$

$\boxed{\text{Series diverges.}}$