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What is on today

1 Ratio test and root test

Briggs-Cochran-Gillett-Schulz §10.7 pp. 696 - 698

Today we will discuss two more tests for convergence. The first is the Ratio Test:

Theorem 1 (Ratio Test)**.** *Let* $\sum a_k$ *be an infinite series, and let*

$$r = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|.$$

1. *If* $r < 1$, *the series converges absolutely, and therefore it converges.*
2. *If* $r > 1$ *(including* $r = \infty$*), the series diverges.*
3. *If* $r = 1$, *the test is inconclusive.*

Example 2 (§10.7 Ex. 10, 14)**.** *Use the Ratio Test to determine whether the following series converge.*

1. $\sum_{k=1}^{\infty} \frac{(-2)^k}{k!}$

2. $\sum_{k=1}^{\infty} \frac{k^k}{2^k}$

Occasionally a series arises for which the preceding tests are difficult to apply. In these situations, we try the Root Test:

Theorem 3 (Root Test). *Let* $\sum a_k$ *be an infinite series and let* $\rho = \lim_{k\to\infty} \sqrt[k]{|a_k|}$.

1. *If* $\rho < 1$, *the series converges absolutely, and therefore it converges.*
2. *If* $\rho > 1$ *(including* $\rho = \infty$*), the series diverges.*
3. *If* $\rho = 1$, *the test is inconclusive.*

Example 4 (§10.7 Ex. 12, 27). *Use the Root Test to determine whether the following series converge.*

1. $\sum_{k=1}^{\infty} \left(-\frac{2k}{k+1}\right)^k$

2. $1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{4}\right)^4 + \cdots$

Table 10.4 Special Series and Convergence Tests

Series or Test	Form of Series	Condition for Convergence	Condition for Divergence	Comments
Geometric series	$\sum_{k=0}^{\infty} ar^k$, $a \neq 0$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	If $\lvert r\rvert < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$.
Divergence Test	$\sum_{k=1}^{\infty} a_k$	Does not apply	$\lim_{k\to\infty} a_k \neq 0$	Cannot be used to prove convergence
Integral Test	$\sum_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and f is continuous, positive, and decreasing	$\int_1^{\infty} f(x)\,dx$ converges.	$\int_1^{\infty} f(x)\,dx$ diverges.	The value of the integral is not the value of the series.
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	Useful for comparison tests
Ratio Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \left\lvert\frac{a_{k+1}}{a_k}\right\rvert < 1$	$\lim_{k\to\infty} \left\lvert\frac{a_{k+1}}{a_k}\right\rvert > 1$	Inconclusive if $\lim_{k\to\infty} \left\lvert\frac{a_{k+1}}{a_k}\right\rvert = 1$
Root Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} < 1$	$\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} > 1$	Inconclusive if $\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} = 1$
Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges.	$b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$, $b_k > 0$	$0 \leq \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges.	$\lim_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0$	$\lim_{k\to\infty} a_k = 0$ and $0 < a_{k+1} \leq a_k$	$\lim_{k\to\infty} a_k \neq 0$	Remainder R_n satisfies $\lvert R_n\rvert \leq a_{n+1}$
Absolute Convergence	$\sum_{k=1}^{\infty} a_k$, a_k arbitrary	$\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges.		Applies to arbitrary series

Now let's try using all of the tests we've learned:

Example 5 (§10.8 Ex. 14, 22, 23, 35, 40, 54)**.** *Determine whether the following series converge. Justify your answers.*

1. $\sum_{k=1}^{\infty} \frac{7k^2-k-2}{4k^4-3k+1}$

2. $\sum_{k=1}^{\infty} \left(\frac{e+1}{\pi}\right)^k$

3. $\sum_{k=1}^{\infty} \frac{k^5}{5^k}$

4. $\sum_{k=1}^{\infty} \frac{2^k 3^k}{k^k}$

5. $\sum_{j=1}^{\infty} \frac{\cos((2j+1)\pi)}{j^2+1}$

6. $\sum_{j=1}^{\infty} j^9 \sin \frac{1}{j^9}$