What is on today

1  Ratio test and root test

1  Ratio test and root test

Briggs-Cochran-Gillett-Schulz §10.7 pp. 696 - 698

Today we will discuss two more tests for convergence. The first is the Ratio Test:

**Theorem 1** (Ratio Test). Let \( \sum a_k \) be an infinite series, and let

\[
    r = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| .
\]

1. If \( r < 1 \), the series converges absolutely, and therefore it converges.

2. If \( r > 1 \) (including \( r = \infty \)), the series diverges.

3. If \( r = 1 \), the test is inconclusive.

**Example 2** (§10.7 Ex. 10, 14). Use the Ratio Test to determine whether the following series converge.

1. \( \sum_{k=1}^{\infty} \frac{(-2)^k}{k!} \)

2. \( \sum_{k=1}^{\infty} \frac{k^k}{2^k} \)
Occasionally a series arises for which the preceding tests are difficult to apply. In these situations, we try the Root Test:

**Theorem 3 (Root Test).** Let \( \sum a_k \) be an infinite series and let \( \rho = \lim_{k \to \infty} \sqrt[|k|]{|a_k|} \).

1. If \( \rho < 1 \), the series converges absolutely, and therefore it converges.
2. If \( \rho > 1 \) (including \( \rho = \infty \)), the series diverges.
3. If \( \rho = 1 \), the test is inconclusive.

**Example 4** (§10.7 Ex. 12, 27). Use the Root Test to determine whether the following series converge.

1. \( \sum_{k=1}^{\infty} \left( -\frac{2k}{k+1} \right)^k \)

2. \( 1 + \left( \frac{1}{2} \right)^2 + \left( \frac{1}{3} \right)^3 + \left( \frac{1}{4} \right)^4 + \cdots \)
Now let’s try using all of the tests we’ve learned:

**Example 5** (§10.8 Ex. 14, 22, 23, 35, 40, 54). **Determine whether the following series converge. Justify your answers.**

1. \( \sum_{k=1}^{\infty} \frac{7k^2-k-2}{4k^3-3k+1} \)
2. $\sum_{k=1}^{\infty} \left( \frac{e+1}{\pi} \right)^k$

3. $\sum_{k=1}^{\infty} \frac{k^5}{3^k}$

4. $\sum_{k=1}^{\infty} \frac{2^k 3^k}{k^k}$
5. \[ \sum_{j=1}^{\infty} \frac{\cos((2j+1)\pi)}{j^2+1} \]

6. \[ \sum_{j=1}^{\infty} j^9 \sin \frac{1}{j^2} \]