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What is on today

1 Ratio test and root test

Briggs-Cochran-Gillett-Schulz §10.7 pp. 696 - 698

Today we will discuss two more tests for convergence. The first is the Ratio Test:

Theorem 1 (Ratio Test)**.** *Let $\sum a_k$ be an infinite series, and let*

$$r = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|. \leftarrow \text{ratio} \geq 0$$

1. *If $r < 1$, the series converges absolutely, and therefore it converges.*
2. *If $r > 1$ (including $r = \infty$), the series diverges.*
3. *If $r = 1$, the test is inconclusive.*

Example 2 (§10.7 Ex. 10, 14)**.** *Use the Ratio Test to determine whether the following series converge.*

1. $\sum_{k=1}^{\infty} \frac{(-2)^k}{k!}$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(-2)^{k+1}}{(k+1)!}\cdot\frac{k!}{(-2)^k}\right|$$

$$= \lim_{k\to\infty}\left|\frac{-2}{k+1}\right| = 0 \Rightarrow \text{series converges by Ratio Test}$$

$$\lim_{k\to\infty}\left|\frac{\frac{(-2)^{k+1}}{(k+1)!}}{\frac{(-2)^k}{k!}}\right| = \lim_{k\to\infty}\left|\frac{(-2)^{k+1}}{(k+1)!}\cdot\frac{k!}{(-2)^k}\right|$$

$$\frac{a}{b} = a\cdot\frac{1}{b}$$

$$= \lim_{k\to\infty}\left|\frac{(-2)^{k+1}}{(-2)^k}\cdot\frac{k!}{(k+1)!}\right| = \lim_{k\to\infty}\left|-2\cdot\frac{\cancel{k!}}{(k+1)\cdot\cancel{k!}}\right| = \lim_{k\to\infty}\left|\frac{-2}{k+1}\right|$$

2. $\sum_{k=1}^{\infty} \frac{k^k}{2^k}$ $= \left(\frac{k}{2}\right)^k$

(Divergence Test also tells us that series diverges since $\lim_{k\to\infty} a_k \neq 0$)

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(k+1)^{k+1}}{2^{k+1}}\cdot\frac{2^k}{k^k}\right| = \left|\frac{(k+1)^{k+1}}{k^k}\cdot\frac{1}{2}\right|$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)^{k+1}}{k^k}\cdot\frac{1}{2}\right| \quad \underline{\text{diverges}}$$

$$\approx \frac{k^{k+1}}{k^k} = k$$

so series diverges by Ratio Test.

by Root Test: $\lim_{k\to\infty}\left(\left(\frac{k}{2}\right)^k\right)^{1/k} = \lim_{k\to\infty}\frac{k}{2}$ diverges, so by Root Test, diverges

Occasionally a series arises for which the preceding tests are difficult to apply. In these situations, we try the Root Test:

Theorem 3 (Root Test). *Let $\sum a_k$ be an infinite series and let $\rho = \lim_{k\to\infty} \sqrt[k]{|a_k|}$.*

1. *If $\rho < 1$, the series converges absolutely, and therefore it converges.*
2. *If $\rho > 1$ (including $\rho = \infty$), the series diverges.*
3. *If $\rho = 1$, the test is inconclusive.*

Example 4 (§10.7 Ex. 12, 27). *Use the Root Test to determine whether the following series converge.*

1. $\sum_{k=1}^{\infty} \left(-\frac{2k}{k+1}\right)^k$

$$\lim_{k\to\infty} \sqrt[\boxed{k}]{|a_k|} = \lim_{k\to\infty} \left(\left|\frac{-2k}{k+1}\right|^{\boxed{k}}\right)^{\boxed{1/k}} = \lim_{k\to\infty} \frac{2k \cdot \frac{1}{k}}{(k+1)\cdot\frac{1}{k}}$$

$$= \lim_{k\to\infty} \frac{2}{1+1/k} = 2 > 1$$

$$\left|\frac{-2k}{k+1}\right| = \frac{2k}{k+1}$$

diverges by the Root Test.

Divergence Test: $\lim_{k\to\infty} \left(\frac{-2k}{k+1}\right)^k = \lim_{k\to\infty} \left(\frac{-2}{1+1/k}\right)^k \neq 0 \Rightarrow$ by Divergence Test also diverges.

2. $1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{4}\right)^4 + \cdots$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^n$$

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{\left(\frac{1}{k}\right)^k} = \lim_{k\to\infty} \left(\frac{1}{k}\right)^{k/k}$$

$$= \lim_{k\to\infty} \left(\frac{1}{k}\right)^1$$

$$= \lim_{k\to\infty} \frac{1}{k} = 0$$

$$a_k = \left(\frac{1}{k}\right)^k$$

$\Rightarrow$ by Root Test it converges

(Divergence Test here would have been inconclusive since $\lim_{k\to\infty} a_k = 0$)

Table 10.4 **Special Series and Convergence Tests**

Series or Test	Form of Series	Condition for Convergence	Condition for Divergence	Comments
Geometric series	$\sum_{k=0}^{\infty} ar^k$, $a \neq 0$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	If $\lvert r\rvert < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$.
Divergence Test	$\sum_{k=1}^{\infty} a_k$	Does not apply	$\lim_{k\to\infty} a_k \neq 0$	Cannot be used to prove convergence
Integral Test	$\sum_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and f is continuous, positive, and decreasing	$\int_1^{\infty} f(x)\,dx$ converges.	$\int_1^{\infty} f(x)\,dx$ diverges.	The value of the integral is not the value of the series. * also a result on remainder
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	Useful for comparison tests
Ratio Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \left\lvert\frac{a_{k+1}}{a_k}\right\rvert < 1$	$\lim_{k\to\infty} \left\lvert\frac{a_{k+1}}{a_k}\right\rvert > 1$	Inconclusive if $\lim_{k\to\infty} \left\lvert\frac{a_{k+1}}{a_k}\right\rvert = 1$
Root Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} < 1$	$\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} > 1$	Inconclusive if $\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} = 1$
Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges.	$b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$, $b_k > 0$	$0 \leq \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges.	$\lim_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0$	$\lim_{k\to\infty} a_k = 0$ and $0 < a_{k+1} \leq a_k$	$\lim_{k\to\infty} a_k \neq 0$	Remainder R_n satisfies $\lvert R_n\rvert \leq a_{n+1}$
Absolute Convergence	$\sum_{k=1}^{\infty} a_k$, a_k arbitrary	$\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges.		Applies to arbitrary series

Now let's try using all of the tests we've learned:

Example 5 (§10.8 Ex. 14, 22, 23, 35, 40, 54). *Determine whether the following series converge. Justify your answers.*

1. $\sum_{k=1}^{\infty} \frac{7k^2-k-2}{4k^4-3k+1}$

Limit Comparison Test

Idea: $\approx \frac{k^2}{k^4} = \frac{1}{k^2}$; use $\sum \frac{1}{k^2}$ to compare

This converges by p-series test with $p=2$.

$$\lim_{k\to\infty} \left(\frac{\frac{7k^2-k-2}{4k^4-3k+1}}{\frac{1}{k^2}} \right) = \lim_{k\to\infty} \frac{(7k^2-k-2)}{4k^4-3k+1} \cdot \frac{k^2}{1}$$

$$= \lim_{k\to\infty} \frac{7k^4 + \cdots}{4k^4 + \cdots} = \frac{7}{4}$$

finite limit, and since $\sum \frac{1}{k^2}$ converges

our series converges by Limit Comparison Test.

2. $\sum_{k=1}^{\infty} \left(\frac{e+1}{\pi}\right)^k$

Root Test

$e \approx 2.7$
$\pi \approx 3.1$

$$\lim_{k\to\infty} \left(\frac{e+1}{\pi}\right)^{k \cdot \frac{1}{k}} = \lim_{k\to\infty} \frac{e+1}{\pi} > 1$$

diverges by Root Test.

OR: divergence test: $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \underbrace{\left(\frac{e+1}{\pi}\right)^k}_{>1} \neq 0$ and so diverges by Divergence Test

OR: geometric series test:
$r = \frac{e+1}{\pi} > 1 \Rightarrow$ diverges as it's a Geometric Series with ratio > 1.

3. $\sum_{k=1}^{\infty} \frac{k^5}{5^k}$

Ratio Test: $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(k+1)^5}{5^{k+1}} \cdot \frac{5^k}{k^5}\right|$

$= \lim_{k\to\infty} \left|\frac{1}{5} \cdot \frac{(k+1)^5}{k^5}\right| = \frac{1}{5} \cdot 1 < 1$

$= \frac{k^5 + \cdots}{k^5} \to 1.$

so converges by Ratio Test

Integral Test: positive ✓
continuous ✓
decreasing ✓

$\int_1^\infty \frac{x^5}{5^x}\,dx \leftarrow$ need to compute this (Integration by Parts)
... might be hard to do this integral.

4. $\sum_{k=1}^{\infty} \frac{2^k 3^k}{k^k}$

Root Test.

$$= \sum_{k=1}^{\infty} \left(\frac{2\cdot 3}{k}\right)^k \qquad \lim_{k\to\infty} \left(\frac{2\cdot 3}{k}\right)^{k/k} = \lim_{k\to\infty} \frac{6}{k} = 0.$$

So converges by Root Test.

5. $\sum_{j=1}^{\infty} \frac{\cos((2j+1)\pi)}{j^2+1}$

$$= \sum_{j=1}^{\infty} \frac{-1}{j^2+1}$$

j	$\cos((2j+1)\pi)$
1	$\cos 3\pi = -1$
2	$\cos 5\pi = -1$
3	$\cos 7\pi = -1$
	$\vdots$

$= \ominus \sum_{j=1}^{\infty} \frac{1}{j^2+1}$; Limit Comparison Test w/ $\sum \frac{1}{j^2}$, which converges by p-Series test w/ $p=2$.

$\lim_{j\to\infty} \frac{\frac{1}{j^2+1}}{\frac{1}{j^2}} = \lim_{j\to\infty} \frac{j^2}{j^2+1} = 1$. so by Limit Comparison Test, since $\sum \frac{1}{j^2}$ converges and the limit is a finite pos. value, the series converges

Comparison Test would also work here since $\frac{1}{j^2+1} < \frac{1}{j^2}$.

6. $\sum_{j=1}^{\infty} j^9 \sin \frac{1}{j^9}$

$$\lim_{j\to\infty} j^9 \cdot \sin\left(\frac{1}{j^9}\right) = \lim_{j\to\infty} \frac{\sin\left(\frac{1}{j^9}\right)}{\frac{1}{j^9}} = \lim_{x\to 0} \frac{\sin(x)}{x} \overset{L'H}{=} \lim_{x\to 0} \frac{\cos(x)}{1} = 1$$

$\left(\frac{1}{1/j^9} = j^9\right)$ $\left(x = \frac{1}{j^9}\right)$

So by Divergence Test, since $\lim_{j\to\infty} a_j \neq 0$, series diverges.