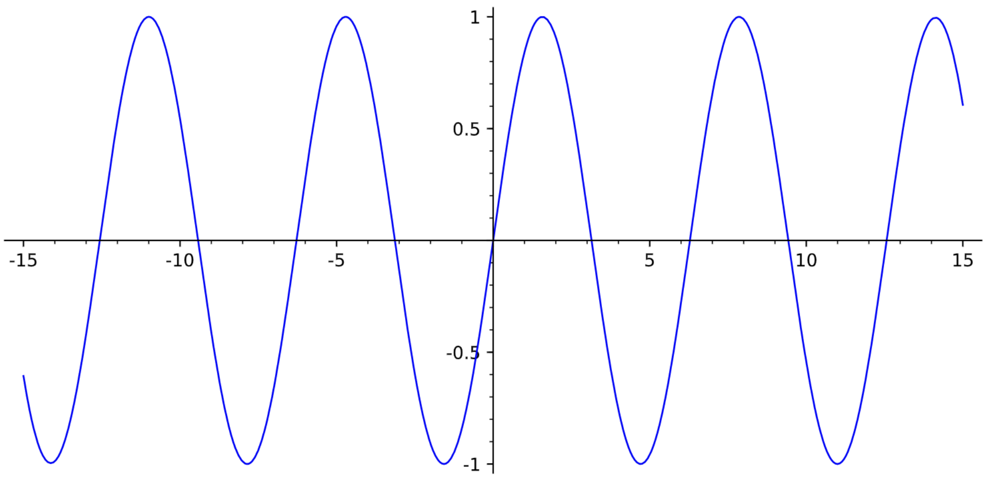
Professor Jennifer Balakrishnan, *jbala@bu.edu*

What is on today

1 Approximating functions with polynomials

Briggs-Cochran-Gillett-Schulz §11.1 pp. 708 - 718

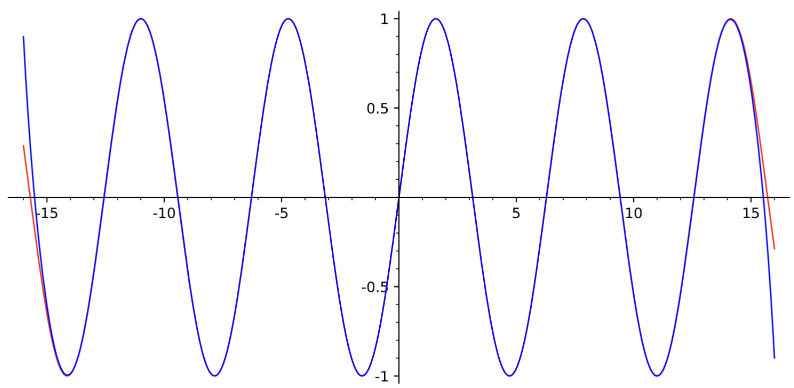
Guess the function!



It's the graph of

$$
\begin{aligned}
y = &-\frac{1}{20397882081197443358640281739902897356800000000}x^{39} + \frac{1}{13763753091226345046315979581580902400000000}x^{37} \\
&- \frac{1}{10333147966386144929666651337523200000000}x^{35} + \frac{1}{8683317618811886495518194401280000000}x^{33} \\
&- \frac{1}{8222838654177922817725562880000000}x^{31} + \frac{1}{8841761993739701954543616000000}x^{29} - \frac{1}{10888869450418352160768000000}x^{27} \\
&+ \frac{1}{15511210043330985984000000}x^{25} - \frac{1}{25852016738884976640000}x^{23} + \frac{1}{51090942171709440000}x^{21} - \frac{1}{121645100408832000}x^{19} \\
&+ \frac{1}{355687428096000}x^{17} - \frac{1}{1307674368000}x^{15} + \frac{1}{6227020800}x^{13} - \frac{1}{39916800}x^{11} + \frac{1}{362880}x^{9} - \frac{1}{5040}x^{7} + \frac{1}{120}x^{5} - \frac{1}{6}x^{3} + x
\end{aligned}
$$

Why does it look like the graph of $y = \sin x$?



($y = \sin x$ is in red)

This is the sort of thing we will investigate today. First, a *power series* is an infinite series of the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots ,$$

or more generally,

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n + \cdots ,$$

where the center of the series a and the coefficients c_k are constants. Another way of thinking about this is that a power series is built up from polynomials of increasing degree:

$$
\begin{aligned}
&c_0 \\
&c_0 + c_1 x \\
&c_0 + c_1 x + c_2 x^2 \\
&\vdots \\
&c_0 + c_1 x + c_2 x^2 + \cdots c_n x^n = \sum_{k=0}^{n} c_k x^k \\
&\vdots \\
&c_0 + c_1 x + c_2 x^2 + \cdots c_n x^n + \cdots = \sum_{k=0}^{\infty} c_k x^k .
\end{aligned}
$$

With this perspective, we begin our exploration of power series by using polynomials to approximate functions.

Earlier, we learned that if a function f is differentiable at a point a, then it can be approximated near a by its tangent line, which is the linear approximation to f at the point a. The linear approximation at a is given by

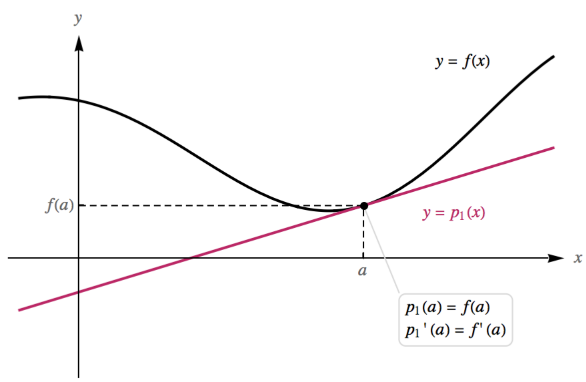
$$y = f(a) + f'(a)(x - a).$$

Because the linear approximation is a first-degree polynomial, we name it p_1:

$$p_1(x) := f(a) + f'(a)(x - a).$$

It matches f in value and in slope at a:

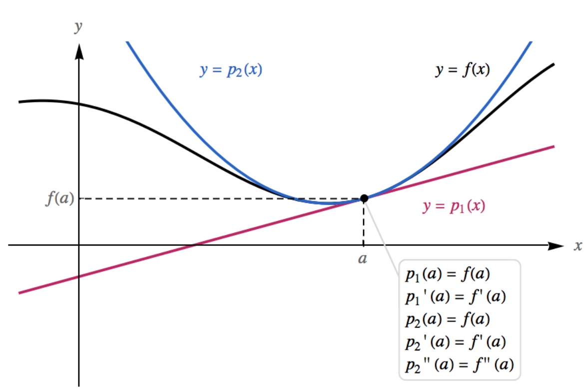
$$p_1(a) = f(a), p_1'(a) = f'(a).$$



Linear approximation works well if f has a fairly constant slope near a. However, if f has a lot of curvature near a, then the tangent line may not provide an accurate approximation. To remedy this situation, we create a quadratic approximating polynomial by adding one new term to the linear polynomial. Denoting this new polynomial p_2, we let

$$p_2(x) := f(a) + f'(a)(x - a) + c_2(x - a)^2 = p_1(x) + c_2(x - a)^2.$$

To determine c_2, we require that p_2 agree with f in value, slope, and concavity at a:



By construction, it already agrees with f in value and slope at a. Differentiating $p_2(x)$ twice and substituting $x = a$ yields

$$p_2''(a) = 2c_2 = f''(a).$$

Thus it follows that $c_2 = \frac{1}{2}f''(a)$ and

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.$$

Example 1 (§11.1 Ex. 10, 12)**.**

(a) *Find the linear approximating polynomial for the following functions centered at the given point a.*

(b) *Find the quadratic approximating polynomial for the following functions centered at the given point a.*

(c) *Use the polynomials obtained in the first two parts to approximate the given quantity.*

1. $f(x) = \frac{1}{x}, a = 1,$ *approximate* $\frac{1}{1.05}$

2. $f(x) = \sqrt{x}, a = 4,$ *approximate* $\sqrt{3.9}$

Assume that f and its first n derivatives exist at a. Our goal is to find an nth-degree polynomial that approximates the values of f near a. The first step is to use p_2 to obtain a cubic polynomial p_3 of the form

$$p_3(x) = p_2(x) + c_3(x-a)^3$$

that satisfies the four matching conditions

$$p_3(a) = f(a), p_3'(a) = f'(a), p_3''(a) = f''(a), p_3'''(a) = f'''(a).$$

Because p_3 is built using p_2, the first three conditions are met. The last condition is used to determine c_3. Differentiating as before, we find $p_3'''(x) = 3 \cdot 2c_3 = 3!c_3$, or in other words,

$$c_3 = \frac{f'''(a)}{3!}.$$

Continuing in this way, building each new polynomial on the previous polynomial, we construct the Taylor polynomials:

Definition 2. *Let f be a function with $f', f'', \dots$ and $f^{(n)}$ defined at a. The nth order Taylor polynomial for f with its center at a, denoted p_n, has the property that it matches f in value, slope, and all derivatives up to the nth derivative at a. That is,*

$$p_n(a) = f(a), p_n'(a) = f'(a), \dots, p_n^{(n)}(a) = f^{(n)}(a).$$

The nth-order Taylor polynomial centered at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Example 3 (§11.1 Ex. 20)**.** *Let $f(x) = \cos x$ and $a = \pi/6$. Find the nth-order Taylor polynomials for $f(x)$ centered at a, for $n = 0, 1, 2$.*

We can look at approximations obtained with Taylor polynomials and give estimates on the remainder in a Taylor polynomial.

Example 4 (§11.1 Ex. 29)**.** *Use a Taylor polynomial of order 2 to approximate* $\sqrt{1.05}$*. Hint: use the quadratic Taylor polynomial approximation of* $f(x) = \sqrt{1+x}$*.*

Taylor polynomials provide good approximations to functions near a specific point. But how accurate are the approximations? To answer this, we define the **remainder** in a Taylor polynomial:

Definition 5. *Let* p_n *be the Taylor polynomial of order* n *for* f*. The remainder in using* p_n *to approximate* f *at the point* x *is* $R_n(x) = f(x) - p_n(x)$*.*

We have the following result quantifying the remainder:

Theorem 6 (Taylor's theorem)**.** *Let* f *have continuous derivatives up to* $f^{(n+1)}$ *on an open interval* I *containing* a*. For all* x *in* I*, we have* $f(x) = p_n(x) + R_n(x)$*, where* p_n *is the* n*th-order Taylor polynomial for* f *centered at* a *and the remainder is*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some point c *between* x *and* a*.*

Example 7 (§11.1 Ex. 43, 44)**.** *Find the remainder* R_n *for the* n*th order Taylor polynomial centered at* a *for the given functions. Express the result for a general value of* n*.*

1. $f(x) = e^{-x}, a = 0.$

2. $f(x) = \cos x, a = \frac{\pi}{2}$.

The difficulty in estimating the remainder is finding a bound for $|f^{(n+1)}(c)|$. Assuming this can be done, we have the following theorem:

Theorem 8 (Estimate of the remainder)**.** *Let n be a fixed positive integer. Suppose there exists a number M such that $|f^{(n+1)}(c)| \leq M$, for all c between a and x, inclusive. The remainder in the nth-order Taylor polynomial for f centered at a satisfies*

$$|R_n(x)| = |f(x) - p_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}.$$

Example 9 (§11.1 Ex. 49)**.** *Use the remainder to find a bound on the error in approximating $e^{0.25}$ with the 4th-order Taylor polynomial centered at 0.*