What is on today

1 Approximating functions with polynomials

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Briggs-Cochran-Gillett-Schulz §11.1 pp. 708 - 718

Guess the function!
It’s the graph of 
\[ y = \frac{1}{20397820811974435864028179902897356800000000} + \frac{1}{10333147966144929866651337523200000000} x^{39} + \frac{1}{1376375309122634504631597958158090240000000000} x^{37} \]
\[ - \frac{1}{822838654177922817725562880000000} x^{31} + \frac{1}{8417619937397015954436160000000} x^{29} - \frac{1}{10888694504183521607680000000} x^{27} \]
\[ + \frac{1}{15512120433095984000000} x^{23} - \frac{1}{25852016738849766400000} x^{21} + \frac{1}{51090942171704944000000} x^{19} \]
\[ - \frac{1}{1216451004088320000} x^{17} + \frac{1}{130767436800000} x^{15} + \frac{1}{39916800} x^{11} + \frac{1}{5040} x^{9} \]
\[ - \frac{1}{6} x^3 + x \]

Why does it look like the graph of \( y = \sin x \)?

\((y = \sin x \text{ is in red})\)

This is the sort of thing we will investigate today. First, a **power series** is an infinite series of the form

\[ \sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots, \]

or more generally,

\[ \sum_{k=0}^{\infty} c_k (x - a)^k = c_0 + c_1 (x - a) + \cdots + c_n (x - a)^n + \cdots, \]

where the center of the series \( a \) and the coefficients \( c_k \) are constants. Another way of thinking about this is that a power series is built up from polynomials of increasing degree:

\[
\begin{align*}
c_0 \\
c_0 + c_1 x \\
c_0 + c_1 x + c_2 x^2 \\
\vdots \\
c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = \sum_{k=0}^{n} c_k x^k \\
\vdots \\
c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots = \sum_{k=0}^{\infty} c_k x^k.
\end{align*}
\]
With this perspective, we begin our exploration of power series by using polynomials to approximate functions.

Earlier, we learned that if a function $f$ is differentiable at a point $a$, then it can be approximated near $a$ by its tangent line, which is the linear approximation to $f$ at the point $a$. The linear approximation at $a$ is given by

$$y = f(a) + f'(a)(x - a).$$

Because the linear approximation is a first-degree polynomial, we name it $p_1$:

$$p_1(x) := f(a) + f'(a)(x - a).$$

It matches $f$ in value and in slope at $a$:

$$p_1(a) = f(a), \quad p_1'(a) = f'(a).$$

Linear approximation works well if $f$ has a fairly constant slope near $a$. However, if $f$ has a lot of curvature near $a$, then the tangent line may not provide an accurate approximation. To remedy this situation, we create a quadratic approximating polynomial by adding one new term to the linear polynomial. Denoting this new polynomial $p_2$, we let

$$p_2(x) := f(a) + f'(a)(x - a) + c_2(x - a)^2 = p_1(x) + c_2(x - a)^2.$$ 

To determine $c_2$, we require that $p_2$ agree with $f$ in value, slope, and concavity at $a$:
By construction, it already agrees with $f$ in value and slope at $a$. Differentiating $p_2(x)$ twice and substituting $x = a$ yields

$$p'_2(a) = 2c_2 = f''(a).$$

Thus it follows that $c_2 = \frac{1}{2} f''(a)$ and

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2.$$  

**Example 1** (**§11.1 Ex. 10, 12**).

(a) Find the linear approximating polynomial for the following functions centered at the given point $a$.

(b) Find the quadratic approximating polynomial for the following functions centered at the given point $a$.

(c) Use the polynomials obtained in the first two parts to approximate the given quantity.

1. $f(x) = \frac{1}{x}, a = 1$, approximate \( f(1.05) \).

   $p_1(x) = f(a) + f'(a)(x-a) = 1 + -1(x-1) = 1-x+1 = 2-x$ \( \approx \) linear approx.

   $p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 = 1 + -1(x-1) + \frac{1}{2}(2)(x-1)^2 = 1-x+1+(x-1)^2 = 2-x+(x-1)^2$ \( \approx \) quadratic approx.

2. $f(x) = \sqrt{x}, a = 4$, approximate $\sqrt{3.9}$.

   $p_1(x) = f(a) + f'(a)(x-a) = f(4) + f'(4)(x-4) = 2 + \frac{1}{2}(x-4)$

   $p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 = f(4) + f'(4)(x-4) + \frac{1}{2} f''(4)(x-4)^2 = 2 + \frac{1}{2}(x-4) + \frac{1}{32} \cdot \frac{1}{2} (x-4)^2 = 2+\frac{1}{2}(x-4)-\frac{1}{64}(x-4)^2$ on a calculator

   $p_1(3.9) = 2+\frac{1}{2}(3.9-4) = 1.975 \approx 1.975$  

   $p_2(3.9) = 1.97484375$  

   $\sqrt{3.9} \approx 1.97484804$
Assume that \( f \) and its first \( n \) derivatives exist at \( a \). Our goal is to find an \( n \)-th-degree polynomial that approximates the values of \( f \) near \( a \). The first step is to use \( p_2 \) to obtain a cubic polynomial \( p_3 \) of the form
\[
p_3(x) = p_2(x) + c_3(x-a)^3
\]
that satisfies the four matching conditions
\[
p_3(a) = f(a), p'_3(a) = f'(a), p''_3(a) = f''(a), p'''_3(a) = f'''(a).
\]
Because \( p_3 \) is built using \( p_2 \), the first three conditions are met. The last condition is used to determine \( c_3 \). Differentiating as before, we find \( p'''_3(x) = 3 \cdot 2c_3 = 3!c_3 \), or in other words,
\[
c_3 = \frac{f'''(a)}{3!}.
\]
Continuing in this way, building each new polynomial on the previous polynomial, we construct the Taylor polynomials:

**Definition 2.** Let \( f \) be a function with \( f', f'', \ldots \) and \( f^{(n)} \) defined at \( a \). The \( n \)-th order Taylor polynomial for \( f \) with its center at \( a \), denoted \( p_n \), has the property that it matches \( f \) in value, slope, and all derivatives up to the \( n \)-th derivative at \( a \). That is,
\[
p_n(a) = f(a), p'_n(a) = f'(a), \ldots, p^{(n)}_n(a) = f^{(n)}(a).
\]
The \( n \)-th-order Taylor polynomial centered at \( a \) is
\[
p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.
\]

**Example 3** (§11.1 Ex. 20). Let \( f(x) = \cos x \) and \( a = \pi/6 \). Find the \( n \)-th-order Taylor polynomials for \( f(x) \) centered at \( a \), for \( n = 0, 1, 2, 5 \).

**6th order Taylor polynomial?**
\[
p_6(x) = f(a) = \frac{\sqrt{3}}{2}
\]

**1st order Taylor polynomial:**
\[
p_1(x) = f(a) + f'(a)(x-a) = \frac{\sqrt{3}}{2} - \frac{1}{2}(x - \frac{\pi}{6})
\]

**2nd order Taylor polynomial:**
\[
p_2(x) = p_1(x) + \frac{f''(a)}{2!}(x-a)^2 = \frac{\sqrt{3}}{2} - \frac{1}{2}(x - \frac{\pi}{6}) + \frac{-\sqrt{3}}{2} \cdot \left(\frac{1}{2}\right)(x - \frac{\pi}{6})^2
\]
\[
p_3(x) = p_2(x) + \frac{f'''(a)}{3!}(x-a)^3
\]
We can look at approximations obtained with Taylor polynomials and give estimates on the remainder in a Taylor polynomial.

**Example 4** (§11.1 Ex. 29). Use a Taylor polynomial of order 2 to approximate $\sqrt{1.05}$. Hint: use the quadratic Taylor polynomial approximation of $f(x) = \sqrt{1+x}$.

**Definition 5.** Let $p_n$ be the Taylor polynomial of order $n$ for $f$. The remainder in using $p_n$ to approximate $f$ at the point $x$ is $R_n(x) = f(x) - p_n(x)$.

We have the following result quantifying the remainder:

**Theorem 6** (Taylor’s theorem). Let $f$ have continuous derivatives up to $f^{(n+1)}$ on an open interval $I$ containing $a$. For all $x$ in $I$, we have $f(x) = p_n(x) + R_n(x)$, where $p_n$ is the $n$th-order Taylor polynomial for $f$ centered at $a$ and the remainder is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1},$$

for some point $c$ between $x$ and $a$.

**Example 7** (§11.1 Ex. 43, 44). Find the remainder $R_n$ for the $n$th order Taylor polynomial centered at $a$ for the given functions. Express the result for a general value of $n$.

1. $f(x) = e^{-x}$, $a = 0$.

$$R_n(x) = \frac{(-1)^{n+1} e^{-c}}{(n+1)!} x^{n+1},$$

for $c$ between 0 and $x$. 

$$f(x) = e^{-x}, \quad f'(x) = -e^{-x}, \quad f''(x) = -e^{-x}, \quad f'''(x) = -e^{-x}, \quad \vdots$$
2. \( f(x) = \cos x, a = \frac{\pi}{2} \).

The difficulty in estimating the remainder is finding a bound for \( |f^{(n+1)}(c)| \). Assuming this can be done, we have the following theorem:

**Theorem 8 (Estimate of the remainder).** Let \( n \) be a fixed positive integer. Suppose there exists a number \( M \) such that \( |f^{(n+1)}(c)| \leq M \), for all \( c \) between \( a \) and \( x \), inclusive. The remainder in the \( n \)th-order Taylor polynomial for \( f \) centered at \( a \) satisfies

\[
|R_n(x)| = |f(x) - p_n(x)| \leq M \frac{|x - a|^{n+1}}{(n+1)!}.
\]

**Example 9** (§11.1 Ex. 49). Use the remainder to find a bound on the error in approximating \( e^{0.25} \) with the 4th-order Taylor polynomial centered at 0.