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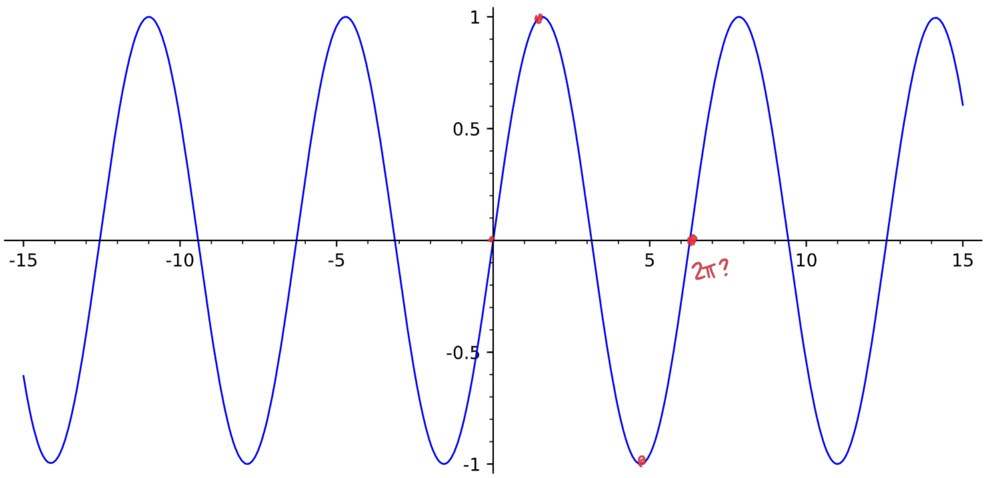
What is on today

1 Approximating functions with polynomials

Briggs-Cochran-Gillett-Schulz §11.1 pp. 708 - 718

Guess the function!

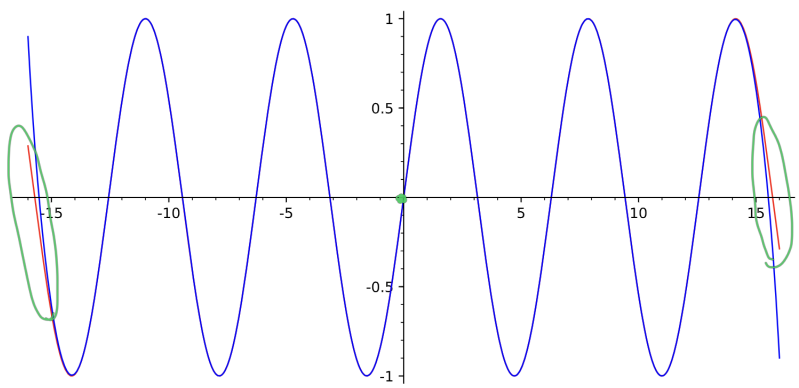


degree 39 polynomial ... looks like $\sin x$

It's the graph of

$$
\begin{aligned}
y = &-\frac{1}{20397882081197443358640281739902897356800000000}x^{39} + \frac{1}{13763753091226345046315979581580902400000000}x^{37} \\
&- \frac{1}{10333147966386144929666651337523200000000}x^{35} + \frac{1}{8683317618811886495518194401280000000}x^{33} \\
&- \frac{1}{8222838654177922817725562880000000}x^{31} + \frac{1}{8841761993739701954543616000000}x^{29} - \frac{1}{10888869450418352160768000000}x^{27} \\
&+ \frac{1}{15511210043330985984000000}x^{25} - \frac{1}{25852016738884976640000}x^{23} + \frac{1}{51090942171709440000}x^{21} - \frac{1}{121645100408832000}x^{19} \\
&+ \frac{1}{355687428096000}x^{17} - \frac{1}{1307674368000}x^{15} + \frac{1}{6227020800}x^{13} - \frac{1}{39916800}x^{11} + \frac{1}{362880}x^{9} - \frac{1}{5040}x^{7} + \frac{1}{120}x^{5} - \frac{1}{6}x^{3} + x
\end{aligned}
$$

Why does it look like the graph of $y = \sin x$?



($y = \sin x$ is in red)

This is the sort of thing we will investigate today. First, a *power series* is an infinite series of the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots ,$$

or more generally,

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n + \cdots ,$$

where the center of the series a and the coefficients c_k are constants. Another way of thinking about this is that a power series is built up from polynomials of increasing degree:

$$
\begin{aligned}
&c_0 \\
&c_0 + c_1 x \\
&c_0 + c_1 x + c_2 x^2 \\
&\vdots \\
&c_0 + c_1 x + c_2 x^2 + \cdots c_n x^n = \sum_{k=0}^{n} c_k x^k \\
&\vdots \\
&c_0 + c_1 x + c_2 x^2 + \cdots c_n x^n + \cdots = \sum_{k=0}^{\infty} c_k x^k .
\end{aligned}
$$

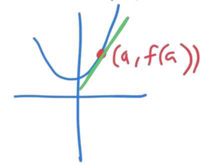
With this perspective, we begin our exploration of power series by using polynomials to approximate functions.

Earlier, we learned that if a function f is differentiable at a point a, then it can be approximated near a by its tangent line, which is the linear approximation to f at the point a. The linear approximation at a is given by

$$y = f(a) + f'(a)(x - a).$$

tangent line approximation

$y = f(a)+f'(a)(x-a)$

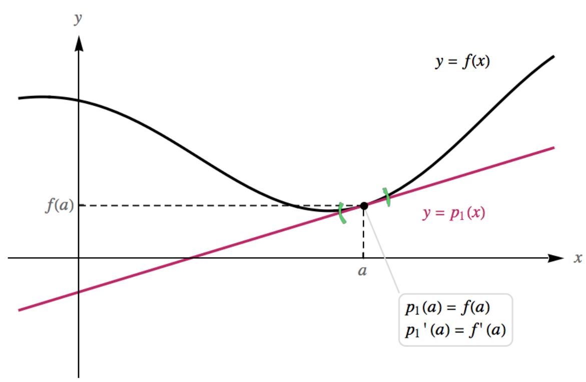


Because the linear approximation is a first-degree polynomial, we name it p_1:

$$p_1(x) := f(a) + f'(a)(x - a).$$

It matches f in value and in slope at a:

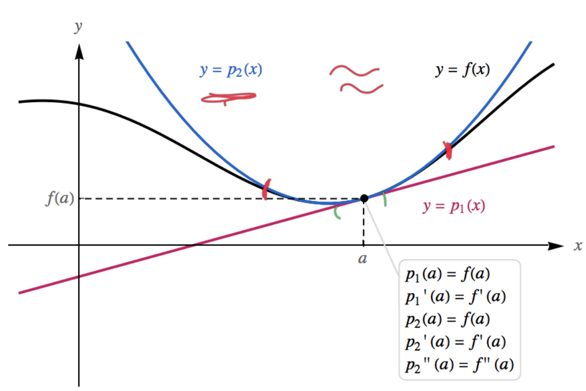
$$p_1(a) = f(a), p_1'(a) = f'(a).$$



Linear approximation works well if f has a fairly constant slope near a. However, if f has a lot of curvature near a, then the tangent line may not provide an accurate approximation. To remedy this situation, we create a quadratic approximating polynomial by adding one new term to the linear polynomial. Denoting this new polynomial p_2, we let

$$p_2(x) := f(a) + f'(a)(x - a) + c_2(x - a)^2 = p_1(x) + c_2(x - a)^2.$$

To determine c_2, we require that p_2 agree with f in value, slope, and concavity at a:



By construction, it already agrees with f in value and slope at a. Differentiating $p_2(x)$ twice and substituting $x = a$ yields

$$p_2''(a) = 2c_2 = f''(a).$$

by construction

$P_2(x) = f(a) + f'(a)(x-a) + c_2(x-a)^2$

$P_2'(x) = f'(a) + 2 \cdot c_2 \cdot (x-a)'$

$P_2''(x) = 2 \cdot c_2 \cdot 1 = \boxed{2c_2 = f''(a)}$

$c_2 = \frac{f''(a)}{2}$.

Thus it follows that $c_2 = \frac{1}{2}f''(a)$ and

$$\boxed{p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.}$$

Example 1 (§11.1 Ex. 10, 12).

(a) *Find the linear approximating polynomial for the following functions centered at the given point a.*

(b) *Find the quadratic approximating polynomial for the following functions centered at the given point a.*

(c) *Use the polynomials obtained in the first two parts to approximate the given quantity.*

1. $\boxed{f(x) = \frac{1}{x}}$, $a = 1$, *approximate* $\boxed{\frac{1}{1.05}}$ so evaluate f at $x = 1.05$

≈ 0.95238

$f(x) = \frac{1}{x} = x^{-1}$ $\quad f(1) = 1$

$f'(x) = -x^{-2}$ $\quad$ @ $a=1$: $f'(1) = -1$

$f''(x) = 2x^{-3}$ $\quad f''(1) = 2.$

$P_1(x) = f(a) + f'(a)(x-a)$

$= 1 + -1(x-1) = 1 - x + 1 = \boxed{2-x}$ ← linear approx.

$P_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$

$= 1 + -1(x-1) + \frac{1}{2}(2)(x-1)^2$

$= 1 - x + 1 + (x-1)^2 = \boxed{2 - x + (x-1)^2}$ ← quadratic approx.

use this to approximate $\frac{1}{1.05}$.

$P_1(1.05) = 2 - 1.05 = \boxed{0.95}$; $\quad P_2(1.05) = 2 - 1.05 + (1.05-1)^2 = 0.9525$

2. $f(x) = \sqrt{x}$, $a = 4$, *approximate* $\sqrt{3.9}$

(3.9 is near 4) so that's why we can use p_1 and p_2 here to approximate $\sqrt{3.9}$

$f(x) = x^{1/2}$

$f'(x) = \frac{1}{2}x^{-1/2}$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2} = -\frac{1}{4}x^{-3/2}$

$P_1(x) = f(a) + f'(a)(x-a)$

$= f(4) + f'(4)(x-4)$

$= 2 + \frac{1}{4}(x-4)$

evaluate at $a = 4$:

$f(4) = 2$

$f'(4) = \frac{1}{2} \cdot \frac{1}{\sqrt{4}} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$f''(4) = -\frac{1}{4} \cdot \frac{1}{4^{3/2}} = -\frac{1}{4} \cdot \frac{1}{2^3} = -\frac{1}{32}$

$p_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

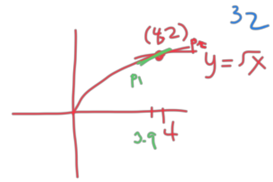
$= f(4) + f'(4)(x-4) + \frac{f''(4)}{2}(x-4)^2$

$= 2 + \frac{1}{4}(x-4) + \frac{-1}{32} \cdot \frac{1}{2}(x-4)^2 = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$

$P_1(3.9) = 2 + \frac{1}{4}(3.9-4) = 1.975$

$P_2(3.9) = 1.97484375$

on a calculator $\sqrt{3.9} \approx 1.97484$



Assume that f and its first n derivatives exist at a. Our goal is to find an nth-degree polynomial that approximates the values of f near a. The first step is to use p_2 to obtain a cubic polynomial p_3 of the form

$$p_3(x) = p_2(x) + c_3(x-a)^3$$

that satisfies the four matching conditions

$$p_3(a) = f(a), p_3'(a) = f'(a), p_3''(a) = f''(a), p_3'''(a) = f'''(a).$$

Because p_3 is built using p_2, the first three conditions are met. The last condition is used to determine c_3. Differentiating as before, we find $p_3'''(x) = 3 \cdot 2c_3 = 3!c_3$, or in other words,

$$c_3 = \frac{f'''(a)}{3!}.$$

Continuing in this way, building each new polynomial on the previous polynomial, we construct the Taylor polynomials:

Definition 2. *Let f be a function with $f', f'', \ldots$ and $f^{(n)}$ defined at a. The nth order Taylor polynomial for f with its center at a, denoted p_n, has the property that it matches f in value, slope, and all derivatives up to the nth derivative at a. That is,*

$$p_n(a) = f(a), p_n'(a) = f'(a), \ldots, p_n^{(n)}(a) = f^{(n)}(a).$$

$c_n = \frac{f^{(n)}(a)}{n!}$

The nth-order Taylor polynomial centered at a is

$$\boxed{p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.}$$

$\pi \simeq 3.14$

Example 3 (§11.1 Ex. 20). *Let $f(x) = \cos x$ and $a = \pi/6$. Find the nth-order Taylor polynomials for $f(x)$ centered at a, for $n = 0, 1, 2$.* $\frac{\pi}{6} \simeq 0.5\ldots$

$\cos(0.6)$?

$f(x) = \cos x$ $a=$ @$\frac{\pi}{6}$: $f(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

$f'(x) = -\sin x$ $f'(\frac{\pi}{6}) = -\frac{1}{2}$

$f''(x) = -\cos x$ $f''(\frac{\pi}{6}) = -\frac{\sqrt{3}}{2}$.

0th order Taylor polynomial?

$p_0(x) = f(a) = \frac{\sqrt{3}}{2}$

$2! = 2 \cdot 1$

1st order Taylor polynomial:

$p_1(x) = f(a) + f'(a)\cdot(x-a) = \frac{\sqrt{3}}{2} + -\frac{1}{2}\left(x - \frac{\pi}{6}\right)$

$= p_0(x) + f'(a)\cdot(x-a)$

2nd order Taylor polynomial:

$p_2(x) = p_1(x) + \frac{f''(a)\cdot(x-a)^2}{2!} = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) + -\frac{\sqrt{3}}{2}\cdot\left(\frac{1}{2}\right)\left(x - \frac{\pi}{6}\right)^2$

$p_3(x) = p_2(x) + \frac{f'''(a)}{3!}(x-a)^3$

We can look at approximations obtained with Taylor polynomials and give estimates on the remainder in a Taylor polynomial.

Example 4 (§11.1 Ex. 29)**.** *Use a Taylor polynomial of order 2 to approximate* $\sqrt{1.05}$*. Hint: use the quadratic Taylor polynomial approximation of* $f(x) = \sqrt{1+x}$*.*

How do we approximate $\sqrt{1.05}$? (Should be close to 1 ... but can we do better??)

approach #1: $f(x) = \sqrt{1+x} = (1+x)^{1/2} = \sqrt{1.05} \Rightarrow x = 0.05$ centered at $a=0$. (to compute $\sqrt{1.05}$, we'd be 0.05 away from our center).

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$ $\qquad f(0) = 1$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-3/2}$ $\qquad f'(0) = \frac{1}{2}$

$f''(0) = -\frac{1}{4}$

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

$$p_2(x) = 1 + \frac{1}{2}(x-0) - \frac{1}{4}\left(\frac{1}{2}\right)(x-0)^2 = 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

$$p_2(0.05) = 1 + \frac{1}{2}(0.05) - \frac{1}{8}(0.05)^2 = 1.0246875$$

approach #2 what about $f(x) = \sqrt{x}$, center at $a=1$.

$\boxed{f(x) = x^{1/2}}$ $\qquad f(1) = 1$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\qquad f'(1) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}x^{-3/2}$ $\qquad f''(1) = -\frac{1}{4}$

$p_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

$= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$

evaluate at 1.05 to approximate $\sqrt{1.05}$

Taylor polynomials provide good approximations to functions near a specific point. But how accurate are the approximations? To answer this, we define the **remainder** in a Taylor polynomial:

Definition 5. *Let* p_n *be the Taylor polynomial of order* n *for* f*. The remainder in using* p_n *to approximate* f *at the point* x *is* $R_n(x) = f(x) - p_n(x)$*.*

We have the following result quantifying the remainder:

examples: $I = (a-1, a+1)$, $I = (a-0.01, a+100)$

Theorem 6 (Taylor's theorem)**.** *Let* f *have continuous derivatives up to* $f^{(n+1)}$ *on an open interval* I *containing* a*. For all* x *in* I*, we have* $f(x) = p_n(x) + R_n(x)$*, where* p_n *is the nth-order Taylor polynomial for* f *centered at* a *and the remainder is*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some point c *between* x *and* a*.*

Example 7 (§11.1 Ex. 43, 44)**.** *Find the remainder* R_n *for the nth order Taylor polynomial centered at* a *for the given functions. Express the result for a general value of* n*.*

1. $f(x) = e^{-x}, a = 0.$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

$f(x) = e^{-x}$

$f'(x) = -e^{-x}$ $\qquad f'''(x) = -e^{-x}$

$f''(x) = -1\cdot(-1)e^{-x} = e^{-x}$ $\qquad f^{(4)}(x) = e^{-x}$

$\vdots$

$f^{(n)}(x) = (-1)^n e^{-x}$

$$R_n(x) = \frac{(-1)^{n+1}e^{-c}}{(n+1)!}x^{n+1}$$

for c between $0, x$.

2. $f(x) = \cos x, a = \frac{\pi}{2}$.

$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

$$\boxed{R_n(x) = \frac{\cos^{(n+1)}(c)\left(x-\frac{\pi}{2}\right)^{n+1}}{(n+1)!},}$$

for c in interval btwn $\frac{\pi}{2}, x$

$f(x) = \cos x$ $a = \frac{\pi}{2}$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{(4)}(x) = \cos x$

The difficulty in estimating the remainder is finding a bound for $|f^{(n+1)}(c)|$. Assuming this can be done, we have the following theorem:

Theorem 8 (Estimate of the remainder)**.** *Let n be a fixed positive integer. Suppose there exists a number M such that $|f^{(n+1)}(c)| \leq M$, for all c between a and x, inclusive. The remainder in the nth-order Taylor polynomial for f centered at a satisfies*

$$|R_n(x)| = |f(x) - p_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}.$$

Example 9 (§11.1 Ex. 49)**.** *Use the remainder to find a bound on the error in approximating $e^{0.25}$ with the 4th-order Taylor polynomial centered at 0.*