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What is on today

1 Approximating functions with polynomials

Briggs-Cochran-Gillett-Schulz §11.1 pp. 708 - 718

Recall from last class the following result quantifying the remainder in Taylor series:

Theorem 1 (Taylor's theorem). *Let f have continuous derivatives up to $f^{(n+1)}$ on an open interval I containing a. For all x in I, we have $f(x) = p_n(x) + R_n(x)$, where p_n is the nth-order Taylor polynomial for f centered at a and the remainder is*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}, \text{ for some point } c \text{ between } x \text{ and } a.$$

The difficulty in estimating the remainder is finding a bound for $|f^{(n+1)}(c)|$. Assuming this can be done, we have the following theorem:

Theorem 2 (Estimate of the remainder). *Let n be a fixed positive integer. Suppose there exists a number M such that $|f^{(n+1)}(c)| \leq M$, for all c between a and x, inclusive. The remainder in the nth-order Taylor polynomial for f centered at a satisfies*

$$|R_n(x)| = |f(x) - p_n(x)| \leq M\frac{|x-a|^{n+1}}{(n+1)!}.$$

Example 3 (§11.1 Ex. 49). *Use the remainder to find a bound on the error in approximating $e^{0.25}$ with the 4th-order Taylor polynomial centered at 0.*

Example 4 (§11.1 Ex. 58)**.** *Consider the approximation* $\sqrt{1+x} \approx 1 + \frac{x}{2}$ *on* $[-0.1, 0.1]$*. Use the remainder to find a bound on the error on the given interval.*

Example 5 (§11.1 Ex. 60)**.** *What is the minimum order of the Taylor polynomial required to approximate* $\sin 0.2$ *with an absolute error no greater than* 10^{-3}*?*

2 Properties of power series

Briggs-Cochran-Gillett-Schulz §11.2 pp. 722 - 726

We saw that Taylor polynomials provide accurate approximations to many functions, and that, in general, the approximations improve as the degree of the polynomials increase. Today we will let the degree of the polynomial increase without bound and produce a power series.

One way to start thinking about power series is to revisit the idea of geometric series: recall that if we fix a real number r such that $|r| < 1$, we have

$$\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \cdots = \frac{1}{1-r}.$$

Now replace the real number r by a variable x. Then we have the following statement:

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots = \frac{1}{1-x}, \text{ if } |x| < 1.$$

This infinite series is an example of a power series, which we define more formally below.

Definition 6. *A power series has the general form*

$$\sum_{k=0}^{\infty} c_k (x-a)^k$$

where a and c_k are real numbers and x is a variable.

- *The c_k are coefficients of the power series and a is the center of the power series.*
- *The set of values of x for which the series converges is its interval of convergence.*
- *The radius of convergence of the power series, denoted R, is the distance from the center of the series to the boundary of the interval of convergence.*

Here is more about the radius of convergence:

Theorem 7. *A power series $\sum_{k=0}^{\infty} c_k (x-a)^k$ centered at a converges in one of three ways:*

1. *The series converges absolutely for all x. It follows that the series converges for all x, in which case the interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.*
2. *There is a real number $R > 0$ such that the series converges absolutely (and therefore converges) for $|x-a| < R$ and diverges for $|x-a| > R$, in which case the radius of convergence is R.*
3. *The series converges only at a, in which case the radius of convergence is $R = 0$.*

We use the Ratio Test or Root Test to determine the interval of convergence for a given power series. We illustrate this in a few examples.

Example 8 (§11.2 Ex. 20, 16, 32)**.** *Determine the radius of convergence of the following power series. Then test the endpoints to determine the interval of convergence.*

1. $\sum_{k=0}^{\infty} \frac{(2x)^k}{k!}$

2. $\sum_{k=0}^{\infty}(-1)^k \frac{x^k}{5^k}$

3. $\sum_{k=0}^{\infty}\left(-\frac{x}{10}\right)^{2k}$

Here is a summary of determining the radius and interval of convergence of the power series $\sum c_k(x-a)^k$:

1. Use the Ratio Test or the Root Test to find the interval $(a-R, a+R)$ on which the series converges absolutely; the radius of convergence for the series is R.
2. Use the *radius* of convergence to find the *interval* of convergence:
   - If $R=\infty$, the interval of convergence is $(-\infty,\infty)$.
   - If $R=0$, the interval of convergence is the single point $x=a$.
   - If $0<R<\infty$, the interval of convergence consists of the interval $(a-R, a+R)$ and possibly one or both of its endpoints. Determining whether the series $\sum c_k(x-a)^k$ converges at the endpoints $x=a-R$ and $x=a+R$ amounts to analyzing the convergence/divergence of the series $\sum c_k(-R)^k$ and $\sum c_k R^k$.

Example 9 (§11.2 Ex. 42, 46)**.** *Use the geometric series*

$$f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

for $|x| < 1$ to find the power series representation for the following functions centered at 0. Give the interval of convergence of the new series.

1. $f(x^3) = \frac{1}{1-x^3}$

2. $f(-4x) = \frac{1}{1+4x}$

Theorem 10 (Combining power series)**.** *Suppose the power series $\sum c_k x^k$ and $\sum d_k x^k$ converge to $f(x)$ and $g(x)$, respectively, on an interval I.*

1. *Sum and difference: The power series $\sum (c_k \pm d_k)x^k$ converges to $f(x) \pm g(x)$ on I.*

2. *Multiplication by a power: Suppose m is an integer such that $k + m \geq 0$ for all terms of the power series $x^m \sum c_k x^k = \sum c_k x^{k+m}$. This series converges to $x^m f(x)$ for all $x \neq 0$ in I. When $x = 0$, the series converges to $\lim_{x\to 0} x^m f(x)$.*

3. *Composition: If* $h(x) = bx^m$*, where* m *is a positive integer and* b *is a nonzero real number, the power series* $\sum c_k(h(x))^k$ *converges to the composite function* $f(h(x))$*, for all* x *such that* $h(x)$ *is in* I*.*

Example 11 (§11.2 Ex. 48)**.** *Use the power series representation*

$$f(x) = \ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$$

for $-1 \leq x < 1$ *to find the power series for the function* $g(x) = x^3 \ln(1-x)$ *centered at 0. Give the interval of convergence of the new series.*