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What is on today

1 Properties of power series

Briggs-Cochran-Gillett-Schulz §11.2 pp. 722 - 726

Theorem 1 (Differentiating and integrating power series). *Suppose the power series* $\sum c_k(x-a)^k$ *converges for* $|x-a| < R$ *and defines a function* f *on that interval.*

1. *Then* f *is differentiable (which implies continuous) for* $|x-a| < R$, *and* f' *is found by differentiating the power series for* f *term by term; that is,* $f'(x) = \sum kc_k(x-a)^{k-1}$, *for* $|x-a| < R$.

2. *The indefinite integral of* f *is found by integrating the power series for* f *term by term; that is,* $\int f(x)dx = \sum c_k \frac{(x-a)^{k+1}}{k+1} + C$, *for* $|x-a| < R$, *where* C *is an arbitrary constant.*

Recap from last time: 1st, find radius of convergence using Root Test or Ratio Test then find interval of convergence by plugging in endpoints.

Example 2 (§11.2 Ex. 52, 54, 56). *Find the power series representation for* g *centered at 0 by differentiating or integrating the power series for* f *(perhaps more than once). Give the interval of convergence for the resulting series.*

1. $g(x) = \frac{1}{(1-x)^3}$ *using* $f(x) = \frac{1}{1-x}$

2 Power series for $f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ for $|x|<1$

$f'(x) = \cancel{0} + 1 + 2x + 3x^2$

Radius of convergence: use Ratio Test:

$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1$

$\lim_{k\to\infty} \left|\frac{(k+1)\cdot \cancel{k} \cdot \cancel{x^{k-1}}\, x}{\cancel{k}\cdot(k-1)\, \cancel{x^{k-2}}}\right| < 1$

$= 1 \cdot \lim_{k\to\infty} |x| < 1$

$= \boxed{|x| < 1}$

put endpts of $x=-1$, $x=1$ into $g(x) = \sum k\cdot(k-1)x^{k-2} \Rightarrow x=-1$

$g(-1) = \sum k(k-1)\cdot(-1)^{k-2} = \sum k(k-1)(-1)^k$

1 How are $f(x)$ and $g(x)$ related?

$f(x) = (1-x)^{-1} \overset{?}{\rightsquigarrow}$ $\boxed{g(x) = (1-x)^{-3} = \frac{f''(x)}{2}}$

$f'(x) = -1(1-x)^{-2}\cdot(-1) = (1-x)^{-2}$

$f''(x) = (-2)(1-x)^{-3}(-1) = 2(1-x)^{-3}$

$f(x) = \sum_{k=0}^{\infty} x^k$

$f'(x) = \sum_{k=1}^{\infty} k\cdot x^{k-1}$

$f''(x) = \sum_{k=2}^{\infty} k\cdot(k-1)\cdot x^{k-2}$

-1, 1 are endpoints of interval

$|x-0| < 1$

3 $g(x) = \frac{f''(x)}{2} = \frac{1}{2}\sum_{k=2}^{\infty} k(k-1)x^{k-2}$

4 radius of convergence:

$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)\cdot k\cdot x^{k-1}}{k\cdot(k-1)\, x^{k-2}}\right| = |x| < 1$ radius of conv. is

5 interval of conv: test $x=-1$, $x=1$.

plug in $x=-1$, $x=1$.

$\frac{1}{2}\sum k(k-1)(-1)^{k-2}$ $\quad \frac{1}{2}\sum k\cdot(k-1)\cdot 1^{k-2}$

diverge.

$\boxed{\text{Interval is } (-1, 1)}$

2. $g(x) = \frac{x}{(1+x^2)^2}$ *using* $f(x) = \frac{1}{1+x^2}$

① How are g and f related?

Recall: $\frac{1}{1-r} = 1+r+r^2+r^3+r^4+\dots = \sum_{k=0}^{\infty} r^k$

$f(x) = (1+x^2)^{-1}$

$f'(x) = (-1)(1+x^2)^{-2}\cdot(2x) = \frac{-2x}{(1+x^2)^2} = -2\cdot g(x).$

$\Rightarrow \boxed{g(x) = -\frac{1}{2}\cdot f'(x)}$

② power series for $f(x) = \frac{1}{1+x^2}$

$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \boxed{\sum_{k=0}^{\infty}(-x^2)^k}$ * could also differentiate this, $\cdot -\frac{1}{2}$ to get series

from here

$f(x) = 1 - x^2 + x^4 - x^6 + x^8 + \dots$

$\Rightarrow \frac{f'(x)}{-2} = \frac{-2x+4x^3-6x^5+8x^7+\dots}{-2}$

$\Rightarrow g(x) = x - 2x^3 + 3x^5 - 4x^7 + \dots$

$\left(= \frac{x}{(1+x^2)^2}\right)$

$= \sum_{k=1}^{\infty}(-1)^k k\, x^{2k-1}$

③ "radius and interval of convergence" = when does this make sense?

(e.g. if $x=10$: $10 - 2\cdot10^3 + 3\cdot10^5 - 4\cdot10^7 + \dots \neq \frac{10}{(1+10^2)^2}$

apply ratio test:

$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)x^{2k+2-1}}{k\cdot x^{2k-1}}\right| < 1$

$= \lim_{k\to\infty}\left|\frac{k+1}{k}\cdot\frac{x^{2k+1}}{x^{2k-1}}\right| < 1$

$= 1\cdot\lim_{k\to\infty}|x^{2k+1-2k+1}| < 1$

$= |x^2| < 1$

$\Rightarrow |x| < 1$

test at $x=-1, x=1$.

plug in, get $\boxed{\text{interval}: (-1,1)}$

3. $g(x) = \ln(1+x^2)$ *using* $f(x) = \frac{x}{1+x^2}$

knew this: $f(x) = \frac{x}{1+x^2} = x\cdot\frac{1}{1+x^2} = x\cdot(1-x^2+x^4-x^6+x^8+\dots) = x\cdot\sum_{k=0}^{\infty}(-x^2)^k$

$= x\sum_{k=0}^{\infty}(-1)^k x^{2k}$

Recall: What is the relationship between f and g?

Found: $g'(x) = 2f(x)$

$\Rightarrow g(x) = \int 2f(x)\,dx$

so once we know series for f, then multiply by 2, integrate to get series for g.

$g(x) = \ln(1+x^2)$

$\boxed{g'(x) = \frac{2x}{1+x^2} = 2\cdot f(x)}$

want: series for $\ln(1+x^2)(= g(x))$

Compute $\int g'(x)\,dx = \int 2\cdot f(x)\,dx$ (know)

$= \int 2(x - x^3 + x^5 - x^7 + x^9 - \dots)\,dx$

$= 2\int x - x^3 + x^5 - x^7 + x^9 - \dots\,dx$

$= 2\cdot\left(\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \frac{x^8}{8} + \frac{x^{10}}{10} - \dots\right) + C$

integrate: $g(x) = \frac{x^2}{1} - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \frac{x^{10}}{5} - \dots + C$

@ $x=0$: $g(0) = \ln(1) = 0 = +C \Rightarrow C = 0$. $\boxed{g(x) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4}\dots}$

$\boxed{\text{interval}: [-1,1]}$

Example 3 (§11.2 Ex, 58, 60)**.** *Find power series representations centered at 0 for the following functions using known power series. Give the interval of convergence for the resulting series.*

1. $f(x) = \frac{1}{1-x^4}$

how does this relate to a function whose power series we already know?

this is close to $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

$\rightsquigarrow$ so take $\frac{1}{1-x^4} = \sum_{k=0}^{\infty}(x^4)^k = \sum_{k=0}^{\infty} x^{4k}$

interval of conv: $\lim_{k\to\infty}\left|\frac{x^{4k+4}}{x^{4k}}\right| < 1$

$= \lim_{k\to\infty}|x^4| < 1 \Rightarrow |x| < 1.$

test endpts: $x=-1$, $x=1$,

find interval: $(-1,1)$.

2. $f(x) = \ln\sqrt{1-x^2}$

Hint: use $\ln(1-x) = -\sum \frac{x^k}{k}$

rewrite $f(x) = \ln\sqrt{1-x^2}$ as $c \cdot \ln(1-x^2)$.

get power series, use ratio test to find interval (check end pts too)

Example 4 (§11.2 Ex. 68)**.** *Find the function represented by the series* $\sum_{k=1}^{\infty} \frac{x^{2k}}{4^k}$ *and find the interval of convergence of the series.*

$\sum_{k=1}^{\infty} \frac{x^{2k}}{4^k} = \sum_{k=1}^{\infty} \left(\frac{x^2}{4}\right)^k$ geometric series with $a = \frac{x^2}{4}$, $r = \frac{x^2}{4}$.

$\frac{\frac{x^2}{4}}{1-\frac{x^2}{4}} = \frac{x^2}{4-x^2}$ is the function

interval of convergence: take $r = \left|\frac{x^2}{4}\right| < 1$

$\Rightarrow |x^2| < 4$

$\Rightarrow -2 < x < 2$.

endpoints are $2, -2$.

check endpts:

at $x=-2$:

$\sum \frac{(-2)^{2k}}{4^k} = \sum \frac{4^k}{4^k}$ diverges

at $x=2$: $\sum \frac{2^{2k}}{4^k} = \sum \frac{4^k}{4^k}$ diverges

$\Leftarrow$ $\boxed{\text{interval is } (-2, 2).}$

2 Taylor series

Briggs-Cochran-Gillett-Schulz §11.3 pp. 731 - 740

Suppose a function f has derivatives $f^{(k)}(a)$ of all orders at the point a. If we write the nth order Taylor polynomial for f centered at a and allow n to increase indefinitely, we get a power series – the Taylor series for f centered at a:

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots = \sum_{k=0}^{\infty} c_k(x-a)^k,$$

where the coefficients are given by

$$c_k = \frac{f^{(k)}(a)}{k!}, \quad k = 0, 1, 2, \ldots .$$

The special case of a Taylor series centered at 0 is called a Maclaurin series.

For the Taylor series to be useful, we need to know two things: the values of x for which the Taylor series converges, and the values of x for which the Taylor series for f *equals* f. We will study the first issue now in a few examples. We will look at the second issue during the next class.

Example 5 (§11.3 Ex. 11, 12). *For each of the following functions,*

(a) *Find the first four nonzero terms of the Maclaurin series.* $\rightsquigarrow a=0$

(b) *Write the power series using summation notation.*

$\boxed{0! = 1}$

(c) *Determine the interval of convergence.*

1. $f(x) = e^{-x} \to +1$ $= \sum_{k=0}^{\infty} c_k (x-a)^k$, $c_k = \dfrac{f^{(k)}(a)}{k!}$

$f'(x) = -e^{-x} \to -1$

$f''(x) = e^{-x} \to +1$

$f'''(x) = -e^{-x} \to -1$

$f^{(4)}(x) = e^{-x} \to +1$

evaluate at $a=0$

$$\sum \frac{c_k \cdot (x-a)^k}{k!} = \frac{1\cdot(x-0)^0}{0!} - \frac{1\cdot(x-0)^1}{1!} + \frac{1\cdot(x-0)^2}{2!} - \frac{1\cdot(x-0)^3}{3!} + \frac{1\cdot(x-0)^4}{4!} \cdots$$

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$$ Maclaurin series

Interval of convergence:

Ratio Test: $\lim_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} \right| = \lim_{k\to\infty} \left| \frac{x \cdot 1}{k+1} \right| = \lim_{k\to\infty} \frac{|x|}{k+1} = 0 < 1$ always true $\Rightarrow$ $(-\infty, \infty)$

2. $f(x) = \cos(2x)$