Professor Jennifer Balakrishnan, *jbala@bu.edu*

What is on today

1 Taylor series

Taylor series approximation to f : $\sum_{k=0}^{\infty} c_k (x-a)^k$, $c_k = \frac{f^{(k)}(a)}{k!}$

Briggs-Cochran-Gillett-Schulz §11.3 pp. 731 - 740

Example 1 (§11.3 Ex. 30, 32)**.** *Find the first four nonzero terms of the Taylor series for the given function centered at a and write the power series using summation notation.*

1. $f(x) = 1/x, a = 2$

$f(x) = \frac{1}{x} = x^{-1}$ $\rightsquigarrow \frac{1}{2}$

$f'(x) = -x^{-2}$ $\rightsquigarrow -\frac{1}{4}$

$f''(x) = 2x^{-3}$ $\rightsquigarrow \frac{2}{8}$

$f'''(x) = -6x^{-4}$ $\rightsquigarrow \frac{-6}{16}$

take these derivatives and evaluate at $a=2$.

Then compute $\sum \frac{f^{(k)}(a)}{k!}(x-a)^k$

$= \frac{1}{2}(x-2)^0 + \left(-\frac{1}{4}\right)\frac{(x-2)^1}{1!} + \frac{1}{4}\frac{(x-2)^2}{2!} - \frac{3}{8}\frac{(x-2)^3}{3!} + \cdots$

$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \cdots$

$= \sum_{k=0}^{\infty} (-1)^k \frac{1}{2^{k+1}}(x-2)^k$

2. $f(x) = e^x, a = \ln 2$

$f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$\vdots$

evaluate at $a = \ln 2$ $\rightsquigarrow$ $f^{(k)}(\ln 2) = e^{\ln 2} = 2$

$\sum_{k=0}^{\infty} \frac{\boxed{f^{(k)}(a)}(x-a)^k}{k!}$ $= \boxed{\sum_{k=0}^{\infty} \frac{2(x-\ln 2)^k}{k!}}$

Here are commonly used Taylor series and the functions to which they converge.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \text{ for } |x| < 1$$

sub $-x$

$\frac{1}{1+x} = \frac{1}{1-(-x)}$

$$\frac{1}{1+x} = 1 - x + x^2 + \cdots + (-1)^k x^k + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k, \text{ for } |x| < 1$$

(could use 1st Taylor series to deduce this one)

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \text{ for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \text{ for } |x| < \infty$$

$(\sin x)' = \cos x$ compute derivative

(if know $\sin x$, can compute series for $\cos x$ by computing derivative of $\sin x$ series)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \text{ for } |x| < \infty$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \text{ for } -1 < x \le 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k}, \text{ for } -1 \le x < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \text{ for } |x| \le 1$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \text{ for } |x| < \infty$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \text{ for } |x| < \infty$$

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k, \text{ for } |x| < 1 \text{ and } \binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, \binom{p}{0} = 1$$

Example 2 (§11.3 Ex. 36, 42)**.** *Use the Taylor series in the table above to find the first four nonzero terms of the Taylor series for the following functions centered at 0.*

1. $\sin x^2$ plug in x^2 into series for $\sin x$: $x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$

2. $x \tan^{-1} x^2$

$\tan^{-1}(x^2) = x^2 - \frac{(x^2)^3}{3} + \frac{(x^2)^5}{5} - \frac{(x^2)^7}{7} + \cdots$

$x \cdot \tan^{-1}(x^2) = x\left(x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \cdots\right) = x^3 - \frac{x^7}{3} + \frac{x^{11}}{5} - \frac{x^{15}}{7} + \cdots$