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What is on today

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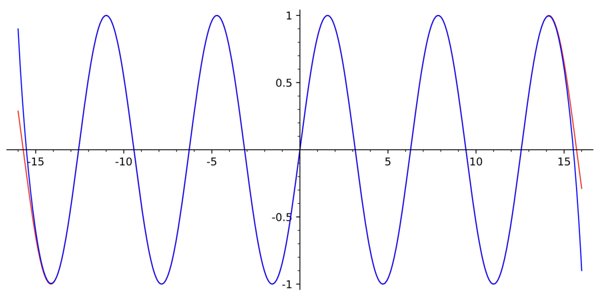
1 Taylor series

Briggs-Cochran-Gillett-Schulz §11.3 pp. 731 - 740

Let's start with a recap of what we've done so far. We've seen that Taylor series can provide a really good approximation to certain functions – remember this?



$$y = -\frac{1}{20397882081197443358640281739902897356800000000}x^{39} + \frac{1}{13763753091226345046315979581580902400000000}x^{37} - \frac{1}{10333147966386144929666651337523200000000}x^{35} + \frac{1}{8683317618811886495518194401280000000}x^{33} - \frac{1}{8222838654177922817725562880000000}x^{31} + \frac{1}{8841761993739701954543616000000}x^{29} - \frac{1}{10888869450418352160768000000}x^{27} + \frac{1}{15511210043330985984000000}x^{25} - \frac{1}{25852016738884976640000}x^{23} + \frac{1}{51090942171709440000}x^{21} - \frac{1}{121645100408832000}x^{19} + \frac{1}{355687428096000}x^{17} - \frac{1}{1307674368000}x^{15} + \frac{1}{6227020800}x^{13} - \frac{1}{39916800}x^{11} + \frac{1}{362880}x^{9} - \frac{1}{5040}x^{7} + \frac{1}{120}x^{5} - \frac{1}{6}x^{3} + x \text{ is in blue}$$

$y = \sin x$ is in red

Where does this come from? We saw that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

is the Taylor series for $\sin x$ centered at $a = 0$. And where does this come from? The Taylor series for a function f centered at a is given by

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots = \sum_{k=0}^{\infty} c_k(x-a)^k,$$

where

$$c_k = \frac{f^{(k)}(a)}{k!}, \quad k = 0, 1, 2, \ldots.$$

Example 1 (§11.3 Ex. 12). *Find the first four nonzero terms of the Taylor series for $f(x) = \cos(2x)$ centered at $a = 0$ and write the power series using summation notation.*

Here are commonly used Taylor series and the functions to which they converge.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \text{ for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 + \cdots + (-1)^k x^k + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k, \text{ for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \text{ for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \text{ for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \text{ for } |x| < \infty$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \text{ for } -1 < x \le 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k}, \text{ for } -1 \le x < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \text{ for } |x| \le 1$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \text{ for } |x| < \infty$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \text{ for } |x| < \infty$$

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k, \text{ for } |x| < 1 \text{ and } \binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, \ \binom{p}{0} = 1$$

We know that if p is a positive integer, then $(1+x)^p$ is a polynomial of degree p; we further have the expansion

$$(1+x)^p = \binom{p}{0} + \binom{p}{1}x + \binom{p}{2}x^2 + \cdots + \binom{p}{p}x^p,$$

where

$$\binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, \quad \binom{p}{0} = 1,$$

for real numbers p and integers $k \geq 1$.

We extend this to functions $f(x) = (1+x)^p$ where p is a nonzero real number:

Theorem 2 (Binomial series)**.** *For real numbers $p \neq 0$, the Taylor series for $f(x) = (1+x)^p$ centered at 0 is the binomial series*

$$\begin{aligned}\sum_{k=0}^{\infty} \binom{p}{k} x^k &= 1 + \sum_{k=1}^{\infty} \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!} x^k \\ &= 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots\end{aligned}$$

The series converges for $|x| < 1$ (and possibly at the endpoints, depending on p). If p is a nonnegative integer, the series terminates and results in a polynomial of degree p.

Example 3 (§11.3 Ex. 47)**.** *Find the first four nonzero terms of the binomial series centered at 0 for $f(x) = \sqrt[4]{1+x}$. Use this to approximate $\sqrt[4]{1.12}$.*

Now we look at when the Taylor series of f actually converges to f on its interval of convergence. Recall that Taylor's Theorem tells us that

$$f(x) = p_n(x) + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

is the remainder, and c is a point between x and a. We see that the remainder $R_n(x) = f(x) - p_n(x)$ measures the difference between f and the approximating Taylor polynomial p_n. When we say that the Taylor series converges to f at a point x, we mean that the value of the Taylor series at x equals $f(x)$: that is, $\lim_{n\to\infty} p_n(x) = f(x)$. We make this precise below.

Theorem 4 (Convergence of Taylor series)**.** *Let f have derivatives of all orders on an open interval I containing a. The Taylor series for f centered at a converges to f, for all x in I, if and only if $\lim_{n\to\infty} R_n(x) = 0$, for all x in I, where*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

is the remainder at x (with c between x and a).

Example 5 (§11.3 Ex. 64, 66)**.** *Find the remainder in the Taylor series centered at the point a for the following functions. Then show that $\lim_{n\to\infty} R_n(x) = 0$ for all x in the interval of convergence.*

1. $f(x) = \cos(2x), a = 0$

2. $f(x) = \cos x, a = \pi/2$

2 Working with Taylor series

Briggs-Cochran-Gillett §11.4 pp. 742 - 747

We now know the Taylor series for many familiar functions, and we have a number of new tools for working with power series. Here we wrap up some additional techniques that make use of what we've studied thus far.

Example 6 (§11.4 Ex. 7, 10, 14)**.** *Evaluate the following limits using Taylor series.*

1. $\lim_{x\to 0} \frac{e^x - 1}{x}$

2. $\lim_{x\to 0} \frac{\sin 2x}{x}$

3. $\lim_{x\to\infty} x \sin \frac{1}{x}$