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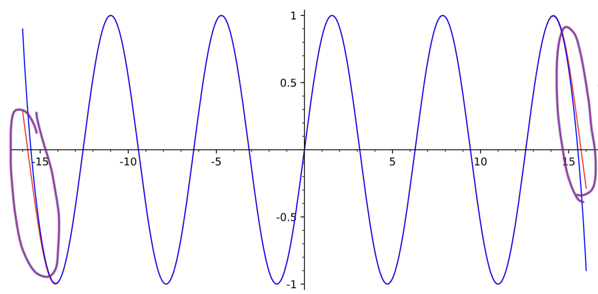
What is on today

1 Taylor series

Briggs-Cochran-Gillett-Schulz §11.3 pp. 731 - 740

Let's start with a recap of what we've done so far. We've seen that Taylor series can provide a really good approximation to certain functions – remember this?



what are these two graphs?

← they look different

$$y = -\frac{1}{20397882081197443358640281739902897356800000000}x^{39} + \frac{1}{137637530912263450463159795815809024000000000}x^{37}$$
$$-\frac{1}{1033314796638614492966665133752320000000}x^{35} + \frac{1}{8683317618811886495518194401280000000}x^{33}$$
$$-\frac{1}{8222838654177922817725562880000000}x^{31} + \frac{1}{8841761993739701954543616000000}x^{29} - \frac{1}{10888869450418352160768000000}x^{27}$$
$$+\frac{1}{15511210043330985984000000}x^{25} - \frac{1}{25852016738884976640000}x^{23} + \frac{1}{51090942171709440000}x^{21} - \frac{1}{121645100408832000}x^{19}$$
$$+\frac{1}{355687428096000}x^{17} - \frac{1}{1307674368000}x^{15} + \frac{1}{6227020800}x^{13} - \frac{1}{39916800}x^{11} + \frac{1}{362880}x^{9} - \frac{1}{5040}x^{7} + \frac{1}{120}x^{5} - \frac{1}{6}x^{3} + x \text{ is in blue}$$

$y = \sin x$ is in red

Where does this come from? We saw that

$$\boxed{\sin x = x - \frac{x^3}{3!}} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

is the Taylor series for $\sin x$ centered at $a = 0$. And where does this come from? The Taylor series for a function f centered at a is given by

$$c_0 + c_1\underline{(x-a)} + c_2\underline{(x-a)}^2 + \cdots + c_n\underline{(x-a)}^n + \cdots = \sum_{k=0}^{\infty} c_k\underline{(x-a)}^k,$$

"centered at $x=a$"

where

$$c_k = \frac{f^{(k)}(a)}{k!}, \quad k = 0, 1, 2, \ldots.$$

Example 1 (§11.3 Ex. 12)**.** *Find the first four nonzero terms of the Taylor series for* $f(x) = \cos(2x)$ *centered at* $a = 0$ *and write the power series using summation notation.*

Taylor series : $\sum_{k=0}^{\infty} c_k (x-a)^k$, $c_k = \frac{f^{(k)}(a)}{k!}$

$f(x) = \cos(2x)$
$f'(x) = -2\sin(2x)$
$f''(x) = -4\cos(2x)$
$f'''(x) = +8\sin(2x)$
$f^{(4)}(x) = 16\cos(2x) = 2^4 \cdot \cos(2x)$
$f^{(5)}(x) = -32\sin(2x)$
$f^{(6)}(x) = -64\cos(2x) = -2^6 \cdot \cos(2x)$

"2k" is our expression for even numbers; (odd numbers: either 2k+1 or 2k−1)

take these derivatives and evaluate at a=0

$\rightsquigarrow f(0) = 1$ — $f(0)\cdot x^0 = 1$
$\rightsquigarrow f'(0) = 0$ ← from eval sin x @ 0 — $\frac{f'(0)\cdot x^1}{1!} = 0$
$\rightsquigarrow f''(0) = -4$ — $\boxed{f''(0)\cdot\frac{x^2}{2!}} = \frac{-4x^2}{2!} = -2x^2$
$\rightsquigarrow f'''(0) = 0$ — $f'''(0)\cdot\frac{x^3}{3!} = 0$
$\boxed{\frac{f^{(4)}(0)\cdot x^4}{4!}} = \frac{16\cdot x^4}{4!} = \frac{16\cdot x^4}{4\cdot 3\cdot 2\cdot 1}$
$\boxed{\frac{f^{(6)}(0)\cdot x^6}{6!}} = \frac{-64\cdot x^6}{6!} = \frac{-64\cdot x^6}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$

← is also going to give

$\Rightarrow 1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} x^{2k}}{(2k)!}$

3·5·3

Here are commonly used Taylor series and the functions to which they converge.

if you know → $$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \text{ for } |x| < 1$$

can get → by substituting $-x$ $\left(\frac{1}{1-(-x)} = \frac{1}{1+x}\right)$
$$\frac{1}{1+x} = 1 - x + x^2 + \cdots + (-1)^k x^k + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k, \text{ for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \text{ for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \text{ for } |x| < \infty$$

can compute $\frac{d}{dx}$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \text{ for } |x| < \infty$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \text{ for } -1 < x \le 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k}, \text{ for } -1 \le x < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \text{ for } |x| \le 1$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \text{ for } |x| < \infty$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \text{ for } |x| < \infty$$

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k, \text{ for } |x| < 1 \text{ and } \binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, \binom{p}{0} = 1$$

We know that if p is a positive integer, then $(1+x)^p$ is a polynomial of degree p; we further have the expansion

$$(1+x)^p = \binom{p}{0} + \binom{p}{1}x + \binom{p}{2}x^2 + \cdots + \binom{p}{p}x^p,$$

where

"p choose k"

$$\frac{p(p-1)(p-2)\cdots 3\cdot 2\cdot 1}{k!\,(p-k)(p-k-1)\cdots 3\cdot 2\cdot 1} = \frac{p!}{k!\,(p-k)!} = \binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}\frac{(p-k)!}{(p-k)!}, \quad \binom{p}{0} = 1, \quad = \frac{p!}{0!\,(p-0)!} = \frac{\cancel{p!}}{0!\,\cancel{p!}} = \frac{1}{0!} = 1 \quad (0! = 1)$$

for real numbers p and integers $k \geq 1$.

We extend this to functions $f(x) = (1+x)^p$ where p is a nonzero real number:

Theorem 2 (Binomial series)**.** *For real numbers $p \neq 0$, the Taylor series for $f(x) = (1+x)^p$ centered at 0 is the binomial series*

$$\sum_{k=0}^{\infty} \binom{p}{k} x^k = 1 + \sum_{k=1}^{\infty} \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!} x^k$$

$$= 1 + px^1 + \frac{\overbrace{p(p-1)}^{\text{2 things to multiply}}}{2!}x^2 + \cdots$$

The series converges for $|x| < 1$ (and possibly at the endpoints, depending on p). If p is a nonnegative integer, the series terminates and results in a polynomial of degree p.

Example 3 (§11.3 Ex. 47)**.** *Find the first four nonzero terms of the binomial series centered at 0 for $f(x) = \boxed{\sqrt[4]{1+x}}$. Use this to approximate $\sqrt[4]{1.12}$.*

$f(x) = (1+x)^{1/4}$

$$\sum_{k=0}^{3} \binom{1/4}{k} x^k = 1 + \binom{\frac14}{1}x^1 + \binom{\frac14}{2}x^2 + \binom{\frac14}{3}x^3$$

$\left(\binom{p}{k}\right.$ is read as "p choose k"$\left.\right)$

$$\binom{\frac14}{1} = \frac14$$

$$\binom{\frac14}{2} = \frac{\frac14\left(\frac14-1\right)}{2!} = \frac{\frac14\left(\frac{-3}{4}\right)}{2} = \frac{-3}{32}$$

$$\binom{\frac14}{3} = \frac{\frac14\left(\frac14-1\right)\left(\frac14-2\right)}{3!} = \frac{\frac14\left(\frac{-3}{4}\right)\left(\frac{-7}{4}\right)}{6\ 2} = \frac{7}{128}$$

$$= 1 + \frac14 x - \frac{3}{32}x^2 + \frac{7}{128}x^3$$

Now to approximate $\sqrt[4]{1.12}$, plug in 0.12 here for x:

$\sqrt[4]{1+x} = \sqrt[4]{1.12}$
$\Rightarrow x = 0.12$

$$1 + \frac14(0.12) - \frac{3}{32}(0.12)^2 + \frac{7}{128}(0.12)^3 \approx \underline{\underline{1.029}}$$

Now we look at when the Taylor series of f actually converges to f on its interval of convergence. Recall that Taylor's Theorem tells us that

$$\underbrace{f(x)}_{} = \underbrace{p_n(x)}_{\text{nth degree Taylor poly}} + R_n(x), \leftarrow \text{nth remainder}$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

is the remainder, and c is a point between x and a. We see that the remainder $R_n(x) = f(x) - p_n(x)$ measures the difference between f and the approximating Taylor polynomial p_n. When we say that the Taylor series converges to f at a point x, we mean that the value of the Taylor series at x equals $f(x)$: that is, $\lim_{n\to\infty} p_n(x) = f(x)$. We make this precise below.

Theorem 4 (Convergence of Taylor series)**.** *Let f have derivatives of all orders on an open interval I containing a. The Taylor series for f centered at a converges to f, for all x in I, if and only if $\lim_{n\to\infty} R_n(x) = 0$, for all x in I, where*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

is the remainder at x (with c between x and a).

Example 5 (§11.3 Ex. 64, 66)**.** *Find the remainder in the Taylor series centered at the point a for the following functions. Then show that $\lim_{n\to\infty} R_n(x) = 0$ for all x in the interval of convergence.*

1. $f(x) = \cos(2x), a = 0$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

$f(x) = \cos(2x) = 2^0 \cos(2x)$

$f'(x) = -2\sin(2x) = -2\sin(2x)$

$f''(x) = -4\cos(2x) = -2^2\cos(2x)$

$f'''(x) = (-4)(-2)\sin(2x) = 2^3 \cdot \sin(2x)$

$f^{(4)}(x) = 8 \cdot 2\cos(2x) = 2^4 \cdot \cos(2x)$

$\vdots$

$(n+1)^{st}$ derivative: $\pm 2^{n+1} \cdot \begin{cases} \cos(2x) \text{ or} \\ \sin(2x) \end{cases}$

$$|R_n(x)| \le \left| \frac{\pm 2^{n+1} \cdot 1}{(n+1)!}(x-0)^{n+1} \right| = \frac{2^{n+1}}{(n+1)!}|x|^{n+1}$$

; now compute $\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!}|x|^{n+1} = 0$

(know that $|\cos(2x)| \le 1$
$|\sin(2x)| \le 1$)

2. $f(x) = \cos x, a = \pi/2$

$$R_n(x) = \frac{f^{(n+1)}(c)\cdot(x-a)^{n+1}}{(n+1)!}$$

$$f^{(k)}(x) = \pm\begin{cases}\cos x \text{ or} \\ \sin x\end{cases}$$

$$= \frac{f^{(n+1)}(c)\left(x-\frac{\pi}{2}\right)^{n+1}}{(n+1)!}$$

$$|R_n(x)| \le \frac{\left|1\cdot\left(x-\frac{\pi}{2}\right)^{n+1}\right|}{(n+1)!}$$

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{\left(x-\frac{\pi}{2}\right)^{n+1}}{(n+1)!} = 0$$

2 Working with Taylor series

Briggs-Cochran-Gillett §11.4 pp. 742 - 747

We now know the Taylor series for many familiar functions, and we have a number of new tools for working with power series. Here we wrap up some additional techniques that make use of what we've studied thus far.

Example 6 (§11.4 Ex. 7, 10, 14)**.** *Evaluate the following limits using Taylor series.*

1. $\lim_{x\to 0} \frac{e^x-1}{x}$ souse Taylor series for e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad (\text{centered at } a=0)$$

$$\lim_{x\to 0} \frac{\left(\cancel{1} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \cancel{1}}{x} = \lim_{x\to 0} \frac{x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots}{x}$$

$$= \lim_{x\to 0} 1 + \cancel{\frac{x}{2} + \frac{x^2}{6} + \cdots}$$

$$= 1$$

2. $\lim_{x\to 0} \frac{\sin 2x}{x}$

$\sin(2x) = ??$

$\boxed{\sin(x)}_{\text{at } a=0} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$\sin(2x) = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots$

$= 2x - \frac{8x^3}{6} + \frac{2^5}{5!}x^5 - \frac{2^7 x^7}{7!} \cdots$

$$= \lim_{x\to 0} \frac{2x - \frac{8x^3}{6} + \frac{2x^5}{5!} - \frac{2^7 x^7}{7!} + \cdots}{x}$$

$$= \lim_{x\to 0} 2 - \cancel{\frac{8x^2}{6}} + \cancel{\frac{2x^4}{5!}} - \cancel{\frac{2^7 x^6}{7!}} + \cdots$$

$$= 2$$

3. $\lim_{x\to\infty} x \sin\frac{1}{x}$

$\sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots$

(let $t = \frac{1}{x}$ (as $x \to \infty$, $t = \frac{1}{x} \to 0$))

$$= \lim_{t\to 0} \frac{1}{t}\sin t = \lim_{t\to 0} \frac{\sin t}{t} = \lim_{t\to 0} \frac{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots}{t}$$

$$= \lim_{t\to 0} 1 - \cancel{\frac{t^2}{3!}} + \cancel{\frac{t^4}{5!}} + \cancel{\cdots} = 1$$