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# What is on today



# 1 Working with Taylor series

Briggs-Cochran-Gillett §11.4 pp. 742 - 747

We wrap up our study of Taylor series today. We now know the Taylor series for many familiar functions, and we have a number of new tools for working with power series. Here we wrap up some additional techniques that make use of what we've studied thus far.

**Example 1** (§11.4 Ex. 16, 22). *Evaluate the following limits using Taylor series.*

1. $\lim_{x\to 4} \frac{x^2-16}{\ln(x-3)}$

Start with

$$\boxed{\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots}$$

(see table from last class)

for $-1 < x \leq 1$

$$= \lim_{x\to 4} \frac{\cancel{(x-4)}(x+4)}{\cancel{(x-4)} - \frac{(x-4)^2}{2} + \frac{(x-4)^3}{3} - \cdots}$$

Idea: shift $x \to x-4$ (change of variables or translation)

$$\ln(x-4+1) = (x-4) - \frac{(x-4)^2}{2} + \frac{(x-4)^3}{3} - \cdots$$

$$\ln(x-4+1) = \ln(x-3)$$

$$= \lim_{x\to 4} \frac{x+4}{1 - \underbrace{\frac{(x-4)}{2}}_{\to 0} + \underbrace{\frac{(x-4)^2}{3}}_{\to 0} - \cdots}$$

all subsequent terms have $(x-4)$ factor, so as $x\to 4$, these $\to 0$.

$$= \frac{8}{1} = 8$$

2. $\lim_{x\to\infty} x(e^{1/x}-1)$

$u = \frac{1}{x} \Rightarrow x = \frac{1}{u}$

$$\boxed{e^u = 1 + \frac{u}{1} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots}$$

$$= \lim_{u\to 0} \frac{1}{u}(e^u - 1)$$

as $x \to \infty$, $u = \frac{1}{x} \to 0$

$$= \lim_{u\to 0} \frac{1}{u}\left(\cancel{1} + u + \frac{u^2}{2} + \frac{u^3}{3!} + \cdots \cancel{-1}\right) = \lim_{u\to 0} 1 + \frac{u}{2} + \frac{u^2}{3!} + \cdots = 1$$

**Example 2** (§11.4 Ex. 31, 32). *For each of the following functions,*

(a) *Differentiate the Taylor series about 0 for the following functions.*

(b) *Identify the function represented by the differentiated series.*

(c) *Give the interval of convergence of the power series for the derivative.*

1. $f(x) = \tan^{-1} x$

$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)$

$= 1 - \frac{3x^2}{3} + \frac{5x^4}{5} - \frac{7x^6}{7} + \cdots$

$= \boxed{1 - x^2 + x^4 - x^6 + \cdots}$ geometric series with ratio $-x^2$

$= \frac{1}{1+x^2} \left( = \frac{1}{1-(-x^2)} \right)$

What is interval of convergence of power series $1 - x^2 + x^4 - x^6 + \cdots$

$|-x^2| < 1 \Rightarrow |x^2| < 1 \Rightarrow |x| < 1$ so we know radius of conv. is 1 to get interval check endpoints, these are $-1$ and $1$:

plug in $\pm 1$: $1 - (\pm 1)^2 + (\pm 1)^4 - (\pm 1)^6 + \cdots$ diverges at each endpt

$\Rightarrow$ $\boxed{\text{interval is } (-1, 1)}$

2. $f(x) = -\ln(1 - x)$

$= x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$, then take $\frac{d}{dx}\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)$

$= 1 + \frac{2x}{2} + \frac{3x^2}{3} + \cdots$

$= 1 + x + x^2 + \cdots$ geometric, ratio is $x$

$= \frac{1}{1-x}$

interval of convergence is $|x| < 1$

$(-1, 1)$

Why converges when $|x| < 1$?

Geometric series with ratio $|r| = |x| < 1$, uses that $\lim_{n\to\infty} |r|^n = 0$

if included $-1$, $1$, $[-1, 1]$ $|x| \leq 1$

**Example 3** (§11.4 Ex. 41)**.** *Use a Taylor series to approximate the definite integral*

$$\int_0^{.35} \tan^{-1} x\, dx.$$

*Use as many terms as needed to ensure the error is less than* $10^{-4}$.

Start with $\tan^{-1}x$ 's Taylor series: $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

$$\int_0^{0.35} \tan^{-1}x\,dx = \int_0^{0.35} x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots dx$$

$$= \frac{x^2}{2} - \frac{x^4}{4\cdot 3} + \frac{x^6}{6\cdot 5} - \frac{x^8}{8\cdot 7} + \cdots \Big|_0^{0.35}$$

$$= \frac{(0.35)^2}{2} - \frac{(0.35)^4}{4\cdot 3} + \frac{(0.35)^6}{6\cdot 5} - \frac{(0.35)^8}{8\cdot 7} + \cdots$$

$\Leftarrow$ this is an alternating series

Error estimate for alternating series: $S - S_n = \boxed{(|R_n| < a_{n+1})}$

in our case: $|R_2| < a_3$

$\Downarrow$

take sum of 2 terms

estimate 3rd term $= \frac{0.35^6}{6\cdot 5} < 10^{-4}$

$= a_{n+1} = a_3 \Rightarrow n = 2$ terms will get you within $10^{-4}$

$\Rightarrow$ So using 2 terms, we compute $\frac{(0.35)^2}{2} - \frac{(0.35)^4}{12} \approx \boxed{0.060}$

**Example 4** (§11.4 Ex. 55, 62)**.** *Identify the functions represented by the following power series.*

1. $\sum_{k=0}^{\infty} \frac{x^k}{2^k} = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k$ this is geometric with ratio $\frac{x}{2}$

$$= \frac{1}{\left(1-\frac{x}{2}\right)}\cdot\frac{2}{2} = \boxed{\frac{2}{2-x}}$$

2. $\sum_{k=1}^{\infty} \frac{x^{2k}}{k} = \sum_{k=1}^{\infty} \frac{(x^2)^k}{k}$

$-\ln(1-x) = 1 + x + x^2 + x^3 + \cdots$

$= \sum \frac{x^k}{k}$

So $\boxed{-\ln(1-x^2)} = \sum \frac{x^{2k}}{k}$

# 2 Review exercises: Taylor series

Briggs-Cochran-Gillett §11.R pp. 750 - 752

**Example 5** (§11.R Ex. 3). *Find the 2nd order Taylor polynomial for $f(x) = \cos^3 x$ centered at $a = 0$.*

$\left( \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \right.$ : Taylor series $\left. \right)$

$f(x) = \cos^3 x$

$f'(x) = 3\cos^2 x(-\sin x) = -3\cos^2 x \sin x$

$f''(x) = -3 \cdot 2\cos x(-\sin x) \cdot \sin x - 3\cos^2 x \cdot \cos x = 6\cos x \sin^2 x - 3\cos^3 x$

Evaluate at $a = 0$

$f(0) = \cos^3(0) = 1$ ✓

$f'(0) = -3\cos^2 0 \sin 0 = 0$

$f''(0) = 6 \cdot \cancel{\cos 0 \cdot \sin^2 0} - 3\cos^3 0 = -3$

$\left( = \frac{f(0)}{0!}(x-0)^0 + \frac{f'(0)(x-0)^1}{1!} + \frac{f''(0)(x-0)^2}{2!} \right)$

$\Rightarrow P_2(x) = 1 + 0 \cdot x - \frac{3}{2} \cdot x^2$

$= \boxed{1 - \frac{3}{2}x^2}$

**Example 6** (§11.R Ex. 14). *Find the remainder term $R_3$ for the Taylor series centered at 0 for the function $f(x) = e^x$. Find an upper bound for the magnitude of this remainder on the interval $|x| < 1$.*

Remainder in Taylor's thm: (Lecture 20)

$R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, for some $c$ between $x, a$

$|R_n(x)| \le M \cdot \frac{|x-a|^{n+1}}{(n+1)!}$, where $|f^{(n+1)}(c)| \le M$

$f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$\vdots$

$f^{(k)}(x) = e^x$

$R_3 = \frac{f^{(4)}(c)(x-0)^4}{4!}$ for some $c$ btwn $x$ and $0$

$|R_3(x)| \le \frac{M \cdot |x|^4}{(n+1)!}$ where $|f^{(4)}(c)| \le M$

$|e^c| \le M$ on $[-1, 1]$

this is the largest when $x = 1$ $\rightarrow e^1 < 3$ Can take $M = 3$

Can take $M = 3$

$R_3(x) \le \frac{3 \cdot 1^4}{4!} = \frac{\cancel{3}}{4 \cdot \cancel{3} \cdot 2 \cdot 1} = \boxed{\frac{1}{8}}$

**Example 7** (§11.R Ex. 18). *Determine the radius and interval of convergence of the power series* $\sum_{k=1}^{\infty} \frac{x^{4k}}{k^2}$.

Use Ratio Test to determine radius of convergence:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{4k+4}}{(k+1)^2}\cdot\frac{k^2}{x^{4k}}\right| = \lim_{k\to\infty}\left|\frac{x^4\cdot \cancel{k^2}}{\cancel{(k+1)^2}}\right| < 1$$

$$|x|^4 < 1 \Rightarrow |x| < 1$$

radius is 1

to determine interval, take the endpoints:

check at $x=-1$: $\sum_{k=1}^{\infty} \frac{(-1)^{4k}}{k^2} = \sum_{k=1} \frac{1}{k^2}$ converges

check at $x=+1$: $\sum_{k=1}^{\infty} \frac{1^{4k}}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges

$\Rightarrow$ interval is $[-1, 1]$

**Example 8** (§11.R Ex. 59). *Use an appropriate Taylor series to find the first four nonzero terms of an infinite series that is equal to* $\sqrt{119}$.

Idea: What is $\sqrt{119}$ close to $\cdots$ $\sqrt{121} = 11$

So look at $f(x) = \sqrt{x}$ centered at $x = 121$

$f(x) = \sqrt{x} = x^{1/2}$

$f'(x) = \frac{1}{2}x^{-1/2}$

$f''(x) = -\frac{1}{4}x^{-3/2}$

$f'''(x) = \frac{3}{8}x^{-5/2}$

evaluate @ center: $\longrightarrow$

$f(121) = \sqrt{121} = 11$

$f'(121) = \frac{1}{2}\cdot 121^{-1/2} = \frac{1}{2}\cdot\frac{1}{11}$

$f''(121) = -\frac{1}{4}\cdot\frac{1}{11^3}$

$f'''(121) = \frac{3}{8}\cdot\frac{1}{11^5}$

Taylor expansion:

$$11 + \frac{1}{22}(x-121) - \frac{1}{4}\cdot\frac{1}{11^3}\cdot\frac{(x-121)^2}{2!} + \frac{3}{8}\cdot\frac{1}{11^5}\frac{(x-121)^3}{3!} + \cdots$$

evaluate at $x = 119$:

$$\Rightarrow 11 + \frac{1}{22}(\underbrace{119-121}_{-2}) - \frac{1}{\cancel{4}\cdot 11^3}\,\frac{\overbrace{(119-121)^2}^{\cancel{2^2}}}{2!} + \frac{\cancel{3}}{\cancel{8}}\cdot\frac{1}{11^5}\,\frac{\overbrace{(119-121)^3}^{-\cancel{2^3}}}{\underbrace{3!}_{\cancel{3}\cdot 2}}$$

$$= \boxed{11 - \frac{1}{11} - \frac{1}{11^3\cdot 2} - \frac{1}{11^5\cdot 2}} \approx \sqrt{119}.$$