Tuesday, April 28, 2020
Final Exam Review

§ 11.2

11.2.21 radius of convergence
interval of convergence of

$$\frac{-x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\frac{x^8}{4!}-\cdots=\sum_{k=1}^{\infty}\frac{(-1)^k x^{2k}}{k!}$$

To compute radius of convergence,
apply Ratio Test:

$$\lim_{k\to\infty}\left|\frac{\cancel{x^{2k+2}}}{\cancel{(k+1)!}_{k+1}}\cdot\frac{\cancel{k!}}{\cancel{x^{2k}}}\right|=\lim_{k\to\infty}\left|\frac{x^2}{k+1}\right|=0\ (\overset{?}{<}1$$

always true

when is it less than 1: here it's true always)

So the radius $R=\infty$ and interval is $(-\infty,\infty)$.

11.2.12 Radius and interval of convergence of

$\sum_{k=1}^{\infty}k!\,(x-10)^k$ Ratio Test:

$$\lim_{k\to\infty}\left|\frac{(k+1)!\cdot(x-10)^{k+1}}{k!\ (x-10)^k}\right|$$

$[=(x-10)\cdot\lim_{k\to\infty}(k+1)]$

$$=\lim_{k\to\infty}\left|\underline{(k+1)}(x-10)\right|=\infty\ (\overset{?}{<}1$$

here it's never true, so $R=0$)

So the radius $R=0$ and the series just converges at the center, so when $x=10$. $\left[\sum_{k=1}^{\infty}k!\,(10-10)^k=\sum 0=0\right]$

When apply Ratio/Root test:

take $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$ or $\lim_{k\to\infty}\sqrt[k]{|a_k|}$

... sometimes get $\left|\begin{smallmatrix}\text{expression}\\ \text{involving}\\ x\end{smallmatrix}\right|<1$ case 1

⇝ this gives some (finite) radius R

(§ 11.2 in textbook)

... sometimes get a number ($\frac{1}{2}$, or 0, or ∞)

Case 2: If number is between 0 and 1, then that is always true

⇝ get $R=\infty$.

.... sometimes get ∞

⇝ $R=0$.

11.4.28

Let $f(x)=\sin(x^2)$.

- Differentiate the Taylor series centered at 0.

- Identify the function (using the series)
- Give interval of convergence of power series for derivative.

From table: $\frac{(x-0)}{n}$ $\frac{(x-0)^3}{n}$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ (center is 0) $\frac{d}{dx}$ (know this + others from table)

How do we get series for $\sin(x^2)$?

$$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots$$

$$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$$

$\boxed{\frac{d}{dx}(\sin x^2) = 2x \cdot \cos(x^2)}$

sanity check: we know should get this at the end.

Now $\frac{d}{dx}\left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots\right)$

$$= 2x - \frac{6 \cdot x^5}{3!} + \frac{10 \cdot x^9}{5!} - \frac{14 \cdot x^{13}}{7!} + \cdots$$

$$= 2x\left(1 - \frac{3x^4}{3!} + \frac{5x^8}{5!} - \frac{7x^{12}}{7!} + \cdots\right)$$

$$= 2x\left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots\right)$$

$$= \boxed{2x\cos(x^2)}$$

Computing Taylor series

Taylor series approximation to $f(x)$ centered at $x=a$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)(x-a)^k}{k!} \qquad (*)$$

Steps:

① Compute f'
f''
f'''
$\vdots$

② Evaluate derivatives at $x=a$: $f(a)$, $f'(a)$, $f''(a)$, $\vdots$

③ Plug this in to the formula $(*)$.

Some of the Taylor series you should know (p. 740 §11.3)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$\frac{1}{1-(-x)}\left(= \frac{1}{1+x}\right) = 1 - x + x^2 - x^3 + \cdots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

can get $\ln(1+x)$ by integration OR $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots$

$-\ln(1-x)$ (use the above with substitution of $x \to -x$ and flip sign)

OR $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

or ① use series for $\frac{1}{1+x^2}$ (get by taking series for $\frac{1}{1+x}$ and sub: $x \to x^2$)

② and then integrate.

$\int 1 - x^2 + x^4 - x^6 \cdots dx$

$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \cdots$

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k \quad \text{where } \binom{p}{k} = \frac{p!}{k!(p-k)!}$$

---

continuing 11.4.28.

$= 2x\left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots\right)$

$= \sum_{k=0}^{\infty} (2x) \frac{(-1)^k x^{4k}}{(2k)!}$

$(2k)(2k-1)\cdots$

$(2k)! \neq 2k!$ $2 \cdot k!$

$\frac{(2k)!}{(2k+2)!} = \frac{1 \cancel{(2k)!}}{(2k+2)(2k+1)\cancel{(2k)!}}$

Now apply Ratio Test:

$$\lim_{k\to\infty} \left| \frac{x^{4k+4}}{(2k+2)!} \cdot \frac{(2k)!}{x^{4k}} \right| = \lim_{k\to\infty} \left| \frac{x^4}{(2k+2)(2k+1)} \right|$$

$= 0 \quad (< 1)$

always less than 1 ... so ∞ for radius

$\Rightarrow$ Radius of convergence is ∞

$\Rightarrow$ Interval of convergence is $(-\infty, \infty)$.

11.4.54

Maclaurin series for $f(x) = \ln(1+x)$ and find Interval of convergence

Evaluate $f(-\frac{1}{2})$ to find value of $\sum_{k=1}^{\infty} \frac{1}{k2^k}$.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$$

Find interval of conv:

Ratio Test: $\lim_{k\to\infty} \left| \frac{x^{k+1}}{k+1} \cdot \frac{k}{x^k} \right| < 1$

$= \lim_{k\to\infty} \left| x \cdot \frac{k}{k+1} \right| < 1$

$= |x| < 1$

Check endpoints: what happens at $x=1$?
" " " $x=-1$

@ $x=1$: $\sum \frac{(-1)^{k+1} \cdot 1}{k}$: alternating harmonic series ⇝ converges

@ $x=-1$: $\sum \frac{(-1)^{k+1}(-1)^k}{k} = \sum \frac{(-1)\cdot(-1)^{2k}}{k}$

$= -\sum \frac{1}{k}$: $-$ (harmonic series) ⇝ diverges

$\Rightarrow$ Interval is $(-1, 1]$

$\ln(1+x) = \sum \frac{(-1)^{k+1} x^k}{k}$

$f(-\frac{1}{2}) = \ln(1 + -\frac{1}{2}) = \sum \frac{\boxed{(-1)^{k+1}} \cdot (\frac{-1}{2})^k}{k}$ $= \frac{(-1)^k}{2^k}$

$= \sum \frac{(-1)^k (-1)\, (-1)^k}{2^k \cdot k}$

$\ln(\frac{1}{2}) = -\sum \frac{(-1)^{2k}}{2^k \cdot k} = -\sum \frac{1}{2^k \cdot k}$

$\ln 1 - \ln 2 = -\ln 2 = -\sum \frac{1}{2^k \cdot k}$ nothing has changed on this side

$\Rightarrow \ln 2 = \sum \frac{1}{2^k \cdot k}$

11.4.61 Identify the function given by power series:

$$\sum_{k=1}^{\infty} \frac{(-1)^k\, k \cdot x^{k+1}}{3^k}$$

(what should we think about when we see k in the numerator?)

Idea: did this come from a derivative ??

$\frac{d}{dx}(x^k) = k \cdot x^{k-1}$

take out x^2 to use exp for $\frac{d}{dx}(x^k)$

$= x^2 \sum_{k=1}^{\infty} \frac{(-1)^k \cdot k \cdot x^{k-1}}{3^k}$

$= x^2 \sum_{k=1}^{\infty} \left(\frac{-1}{3}\right)^k \cdot k x^{k-1}$

$= x^2 \sum_{k=1}^{\infty} \left(\frac{-1}{3}\right)^k \frac{d}{dx}(x^k)$

$$= x^2 \sum_{k=1} \frac{d}{dx}\left(\left(-\frac{x}{3}\right)^k\right)$$

(Idea: $\left(\frac{-}{3}\right)$ is a constant, doesn't have x ... so can move it)

$$= x^2 \frac{d}{dx}\left(\sum_{k=1}^{\infty}\left(\frac{-x}{3}\right)^k\right)$$

(sum of derivatives is the same as the derivative of sum)

$= \frac{d}{dx}(c_1 x + c_2 x^2 + \dots)$

$$= x^2 \frac{d}{dx}\left(\sum_{k=0}^{\infty}\left(\frac{-x}{3}\right)^k\right)$$

$= \frac{d}{dx}(c_0 + c_1 x + c_2 x^2 + \dots)$ since $\frac{d}{dx}(c_0) = 0$.

$$= x^2 \frac{d}{dx}\left(\frac{1}{1-\left(\frac{-x}{3}\right)}\right) = x^2 \cdot \frac{d}{dx}\left(\frac{3}{3+x}\right)$$

$$= x^2 \cdot 3 \cdot (3+x)^{-2} \cdot (-1)$$

$$= \boxed{\frac{-3x^2}{(x+3)^2}}$$