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What is on today

1 Final exam review II

Here are some techniques to compute volumes:

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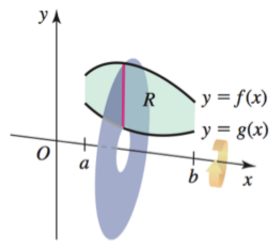
Integration with respect to x



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Disk/washer method about the x–axis

Disks/washers are *perpendicular* to the x–axis.

$$\int_a^b \pi\left(f(x)^2 - g(x)^2\right)dx$$

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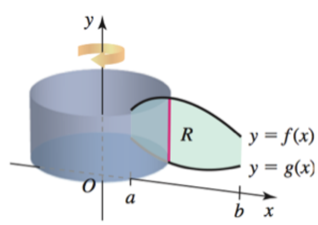

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Shell method about the y–axis

Shells are *parallel* to the y–axis.

$$\int_a^b 2\pi x\,(f(x) - g(x))\,dx$$

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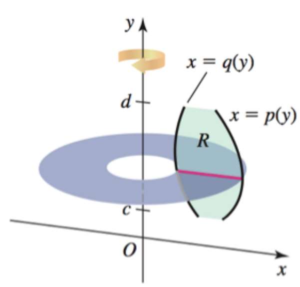
Integration with respect to y



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Disk/washer method about the y–axis

Disks/washers are *perpendicular* to the y–axis.

$$\int_c^d \pi\left(p(y)^2 - q(y)^2\right)dy$$

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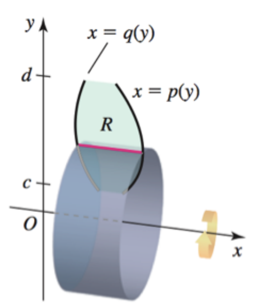

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Shell method about the x–axis

Shells are *parallel* to the x–axis.

$$\int_c^d 2\pi y\,(p(y) - q(y))\,dy$$

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Here is a table of commonly used antiderivatives:

1. $\int k\,dx = kx + C$, k real

2. $\int x^p\,dx = \frac{x^{p+1}}{p+1} + C$, $p \neq -1$ real

3. $\int \cos ax\,dx = \frac{1}{a}\sin ax + C$

4. $\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$

5. $\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$

6. $\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C$

7. $\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C$

8. $\int \csc ax \cot ax\,dx = -\frac{1}{a}\csc ax + C$

9. $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

10. $\int \frac{dx}{x} = \ln|x| + C$

11. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

12. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$, $a > 0$

13. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$, $a > 0$

14. $\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$

15. $\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$

16. $\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$

17. $\int \csc ax\,dx = -\frac{1}{a}\ln|\csc ax + \cot ax| + C$

Example 1 (§8.5.67)**.** *Find the area of the region bounded by the curve* $y = \frac{10}{x^2 - 2x - 24}$*, the* x*-axis, and the lines* $x = -2$ *and* $x = 2$*.*

Here is a table of all convergence tests:

Table 10.4 Special Series and Convergence Tests

Series or Test	Form of Series	Condition for Convergence	Condition for Divergence	Comments
Geometric series	$\sum_{k=0}^{\infty} ar^k,\ a \neq 0$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	If $\lvert r\rvert < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$.
Divergence Test	$\sum_{k=1}^{\infty} a_k$	Does not apply	$\lim_{k\to\infty} a_k \neq 0$	Cannot be used to prove convergence
Integral Test	$\sum_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and f is continuous, positive, and decreasing	$\int_1^{\infty} f(x)\,dx$ converges.	$\int_1^{\infty} f(x)\,dx$ diverges.	The value of the integral is not the value of the series.
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$	Useful for comparison tests
Ratio Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \left\lvert\frac{a_{k+1}}{a_k}\right\rvert < 1$	$\lim_{k\to\infty} \left\lvert\frac{a_{k+1}}{a_k}\right\rvert > 1$	Inconclusive if $\lim_{k\to\infty} \left\lvert\frac{a_{k+1}}{a_k}\right\rvert = 1$
Root Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} < 1$	$\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} > 1$	Inconclusive if $\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} = 1$
Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges.	$b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0, b_k > 0$	$0 \leq \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges.	$\lim_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0$	$\lim_{k\to\infty} a_k = 0$ and $0 < a_{k+1} \leq a_k$	$\lim_{k\to\infty} a_k \neq 0$	Remainder R_n satisfies $\lvert R_n\rvert \leq a_{n+1}$
Absolute Convergence	$\sum_{k=1}^{\infty} a_k$, a_k arbitrary	$\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges.		Applies to arbitrary series

Example 2 (Midterm 2 #8f, #8h)**.** *Do the following converge or diverge?*

1. $\sum_{k=3}^{\infty} \frac{2}{k(2-k)}$

2. $\left\{\frac{(3k^2+2k+1)\sin(k)}{4k^3+k}\right\}_{k=1}^{\infty}$

Here is a list of commonly used Taylor series:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \text{ for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 + \cdots + (-1)^k x^k + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k, \text{ for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \quad \text{for } -1 < x \le 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k}, \quad \text{for } -1 \le x < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad \text{for } |x| \le 1$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \text{ for } |x| < \infty$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \text{ for } |x| < \infty$$

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k, \quad \text{for } |x| < 1 \text{ and } \binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, \quad \binom{p}{0} = 1$$

Example 3 (§11.4 Ex. 33)**.** *Consider the differential equation $y'(t) - y = 0$ with initial condition $y(0) = 2$. Find a power series for the solution of the differential equation and identify the function represented by the power series.*