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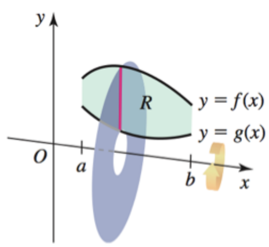
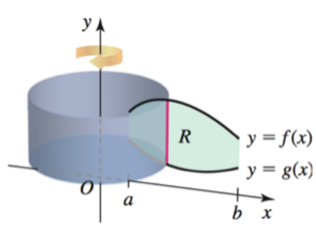
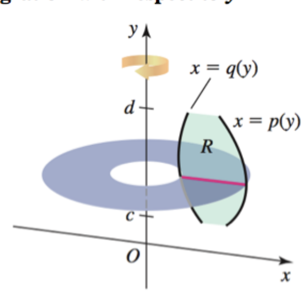
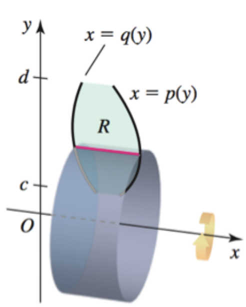
What is on today

1 Final exam review II

Here are some techniques to compute volumes:

Integration with respect to x  	**Disk/washer method about the x-axis** Disks/washers are *perpendicular* to the x-axis. $\int_a^b \pi\left(f(x)^2 - g(x)^2\right) dx$ (note that disk/washer around x-axis gives integral in terms of x)
 	**Shell method about the y-axis** Shells are *parallel* to the y-axis. $\int_a^b 2\pi x\,(f(x) - g(x))\, dx$ (note that shells around y-axis give integral in terms of x)
Integration with respect to y  	**Disk/washer method about the y-axis** Disks/washers are *perpendicular* to the y-axis. $\int_c^d \pi\left(p(y)^2 - q(y)^2\right) dy$
 	**Shell method about the x-axis** Shells are *parallel* to the x-axis. $\int_c^d 2\pi y\,(p(y) - q(y))\, dy$

Here is a table of commonly used antiderivatives:

1. $\int k\,dx = kx + C$, k real

2. $\int x^p\,dx = \frac{x^{p+1}}{p+1} + C$, $p \neq -1$ real

3. $\int \cos ax\,dx = \frac{1}{a}\sin ax + C$

4. $\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$

5. $\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$

6. $\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C$

7. $\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C$

8. $\int \csc ax \cot ax\,dx = -\frac{1}{a}\csc ax + C$

9. $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

10. $\int \frac{dx}{x} = \ln|x| + C$

11. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

12. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$, $a > 0$

13. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$, $a > 0$

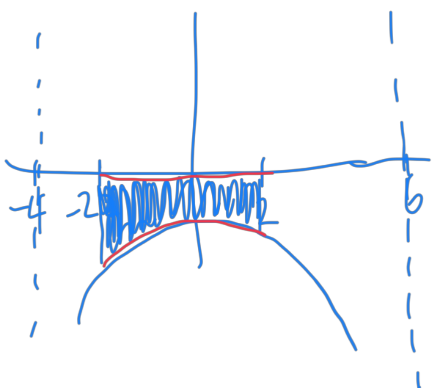
14. $\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$

15. $\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$

16. $\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$

17. $\int \csc ax\,dx = -\frac{1}{a}\ln|\csc ax + \cot ax| + C$

Example 1 (§8.5.67). *Find the area of the region bounded by the curve* $y = \frac{10}{x^2-2x-24}$, *the* x*-axis, and the lines* $x = -2$ *and* $x = 2$.



$$y = \frac{10}{x^2-2x-24} = \frac{10}{(x+4)(x-6)}$$

area:

$$\int_{-2}^{2} 0 - \frac{10}{x^2-2x-24}\,dx$$

$$= \int_{-2}^{2} -\frac{10}{(x+4)(x-6)}\,dx$$

test if graph is pos/neg between $x=-4$ and 6: plug in something in the interval, e.g. $x=0$.

Now use partial fractions: $\frac{10}{(x+4)(x-6)} = \frac{A}{x+4} + \frac{B}{x-6}$

$$\Rightarrow 10 = A(x-6) + B(x+4)$$

$$10 = Ax - 6A + Bx + 4B$$

$$\Rightarrow -6A + 4B = 10$$

$$+)\ 6(A\cancel{x} + B\cancel{x} = 0\cancel{x})$$

$$10B = 10 \Rightarrow B = 1$$

$$\Rightarrow A = -1$$

$$\Rightarrow \frac{10}{(x+4)(x-6)} = \frac{-1}{x+4} + \frac{1}{x-6}.$$

$$\int_{-2}^{2} 0 - \left(\frac{-1}{x+4} + \frac{1}{x-6}\right)dx$$

$$= \int_{-2}^{2} \frac{1}{x+4} - \frac{1}{x-6}\,dx = \ln|x+4| - \ln|x-6|\Big|_{-2}^{2}$$

$$= \ln(6) - \ln(|-4|) - (\ln 2 - \ln(|-8|))$$

$$= \ln 6 - \ln 4 - \ln 2 + \ln 8 = \ln\frac{(6\cdot 8)}{(4\cdot 2)} = \ln 6$$

Here is a table of all convergence tests:

Table 10.4 Special Series and Convergence Tests

Series or Test	Form of Series	Condition for Convergence	Condition for Divergence	Comments
Geometric series	$\sum_{k=0}^{\infty} ar^k,\ a \neq 0$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	If $\lvert r\rvert < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$.
Divergence Test	$\sum_{k=1}^{\infty} a_k$	Does not apply	$\lim_{k\to\infty} a_k \neq 0$	Cannot be used to prove convergence
Integral Test	$\sum_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and f is continuous, positive, and decreasing	$\int_1^{\infty} f(x)\,dx$ converges.	$\int_1^{\infty} f(x)\,dx$ diverges.	The value of the integral is not the value of the series. * Remainder result here as well – see book/notes
p-series	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	$p > 1$	$p \leq 1$ (p=1 is harmonic)	Useful for comparison tests
Ratio Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \left\lvert\frac{a_{k+1}}{a_k}\right\rvert < 1$	$\lim_{k\to\infty} \left\lvert\frac{a_{k+1}}{a_k}\right\rvert > 1$	Inconclusive if $\lim_{k\to\infty} \left\lvert\frac{a_{k+1}}{a_k}\right\rvert = 1$
Root Test	$\sum_{k=1}^{\infty} a_k$	$\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} < 1$	$\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} > 1$	Inconclusive if $\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} = 1$
Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0$	$a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges.	$b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$, where $a_k > 0,\ b_k > 0$	$0 \leq \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges.	$\lim_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges.	$\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$.
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0$	$\lim_{k\to\infty} a_k = 0$ and $0 < a_{k+1} \leq a_k$	$\lim_{k\to\infty} a_k \neq 0$	Remainder R_n satisfies $\lvert R_n\rvert \leq a_{n+1}$
Absolute Convergence	$\sum_{k=1}^{\infty} a_k$, a_k arbitrary	$\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges.		Applies to arbitrary series

Example 2 (Midterm 2 #8f, #8h). *Do the following converge or diverge?*

1. $\sum_{k=3}^{\infty} \frac{2}{k(2-k)} = \frac{2}{2k-k^2}$

(idea: use limit comparison with $\frac{1}{k^2}$: we know $\sum_{k=3}^{\infty} \frac{1}{k^2}$ converges by p-series test since $p=2$)

To use limit comparison, need to have series w/ pos terms consider $\frac{-2}{2k-k^2} = \frac{2}{k^2-2k}$

$$\lim_{k\to\infty} \frac{\frac{-2}{2k-k^2}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{-2k^2}{2k-k^2} = +2 \Rightarrow \sum \frac{-2}{2k-k^2} \text{ converges}$$

by limit comparison test with $\sum \frac{1}{k^2}$ $\Rightarrow \sum \frac{2}{2k-k^2}$ converges as well.

2. $\left\{\frac{(3k^2+2k+1)\sin(k)}{4k^3+k}\right\}_{k=1}^{\infty}$

1) temp. ignore $\sin(k)$: $\lim_{k\to\infty} \frac{3k^2+2k+1}{4k^3+k} = 0$

2) know $\lvert\sin(k)\rvert \leq 1$

$$\frac{(3k^2+2k+1)(-1)}{4k^3+k} \leq \frac{(3k^2+2k+1)\cdot\sin k}{4k^3+k} \leq \frac{3k^2+2k+1}{4k^3+k}$$

Now take $\lim_{k\to\infty}$ of everything

$$\Rightarrow 0 \leq \lim_{k\to\infty} \frac{(3k^2+2k+1)(\sin k)}{4k^3+k} \leq 0$$

Sequence: $\{a_1, a_2, a_3, a_4, \dots\}$

So the sequence converges $\Rightarrow 0$

Here is a list of commonly used Taylor series:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \text{ for } |x| < 1$$ ✓

use for this

$$\frac{1}{1+x} = 1 - x + x^2 + \cdots + (-1)^k x^k + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k, \text{ for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \text{ for } |x| < \infty$$ ✓

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \text{ for } |x| < \infty$$ ✓

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \text{ for } |x| < \infty$$ use for this

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \text{ for } -1 < x \le 1$$ then sub.

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k}, \text{ for } -1 \le x < 1$$ by integrating $\frac{1}{1-x}$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \text{ for } |x| \le 1$$ use series for $\frac{1}{1+x^2}$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \text{ for } |x| < \infty$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \text{ for } |x| < \infty$$

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k, \text{ for } |x| < 1 \text{ and } \binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, \ \binom{p}{0} = 1$$ ✓

Example 3 (§11.4 Ex. 33). *Consider the differential equation* $\boxed{y'(t) - y = 0}$ *with initial condition* $y(0) = 2$. *Find a power series for the solution of the differential equation and identify the function represented by the power series.*

$y(t) = \sum_{t=0}^{\infty} c_k t^k$ (this is a power series)

$c_k = \frac{y^{(k)}(0)}{k!}$

$2 = y(0) = c_0 + c_1 \cancel{t} + c_2 \cancel{t}^2 + \dots \Big|_{t=0}$

$\Rightarrow c_0 = 2.$

now use diff. eq: $y'(t) = y(t)$

$y'(0) = y(0) = 2 = c_1 = \frac{y'(0)}{1!} = 2.$

$y'(t) = y(t)$ ↓ differentiate both sides

$\Rightarrow y''(t) = y'(t)$

$\Rightarrow y'''(t) = y''(t)$ $\Rightarrow$ $y''(0) = y'''(0) = \dots = y^{(k)}(0) = 2.$

$\Rightarrow c_2 = \frac{2}{2!}, \quad c_3 = \frac{2}{3!}, \quad c_4 = \frac{2}{4!} \dots$

$\Rightarrow y(t) = 2 + 2t + \frac{2t^2}{2!} + \frac{2t^3}{3!} + \dots$

$= 2\left(1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \dots\right) = 2 \cdot e^t$

Sanity check

$y'(t) = y(t)$

$\frac{dy}{dt} = y$

$\int \frac{dy}{y} = \int dt$

$\ln|y| = t + C$

$y = C' e^t$

use $y(0) = 2$

$\Rightarrow y = 2e^t$

polynomial long division

e.g.: $\frac{x^2 - 1}{x+2}$

$$\begin{array}{r} x - 2 + \frac{3}{x+2} \\ x+2 \,\overline{)\, x^2 + 0x - 1} \\ -)\ x^2 + 2x \quad\ \ \\ \hline -2x - 1 \\ -)\ -2x - 4 \\ \hline 3 \end{array}$$

$= \boxed{x - 2 + \frac{3}{x+2}}$

To do:

- Remainders of series
- Volumes of regions rotated around lines parallel to axes