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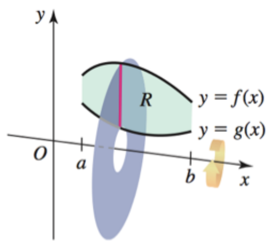
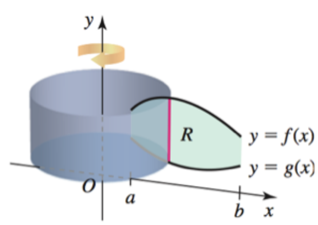
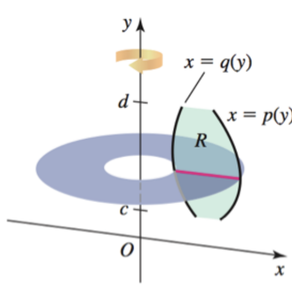
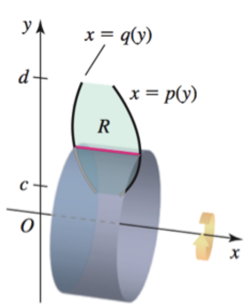
# What is on today



# 1 Final exam review II

Here are some techniques to compute volumes:

| | |
|---|---|
| **Integration with respect to $x$**<br><br> | **Disk/washer method about the $x$–axis**<br>Disks/washers are *perpendicular* to the $x$–axis.<br>$\int_a^b \pi\left(f(x)^2 - g(x)^2\right)dx$ |
| <br> | **Shell method about the $y$–axis**<br>Shells are *parallel* to the $y$–axis.<br>$\int_a^b 2\pi x\,(f(x) - g(x))\,dx$ |
| **Integration with respect to $y$**<br><br> | **Disk/washer method about the $y$–axis**<br>Disks/washers are *perpendicular* to the $y$–axis.<br>$\int_c^d \pi\left(p(y)^2 - q(y)^2\right)dy$ |
| <br> | **Shell method about the $x$–axis**<br>Shells are *parallel* to the $x$–axis.<br>$\int_c^d 2\pi y\,(p(y) - q(y))\,dy$ |

Here is a table of commonly used antiderivatives:

**1.** $\int k\,dx = kx + C$, $k$ real

**2.** $\int x^p\,dx = \frac{x^{p+1}}{p+1} + C$, $p \neq -1$ real

**3.** $\int \cos ax\,dx = \frac{1}{a}\sin ax + C$

**4.** $\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$

**5.** $\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$

**6.** $\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C$

**7.** $\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C$

**8.** $\int \csc ax \cot ax\,dx = -\frac{1}{a}\csc ax + C$

**9.** $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

**10.** $\int \frac{dx}{x} = \ln|x| + C$

**11.** $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

**12.** $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$, $a > 0$

**13.** $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$, $a > 0$

**14.** $\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$

**15.** $\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$

**16.** $\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$

**17.** $\int \csc ax\,dx = -\frac{1}{a}\ln|\csc ax + \cot ax| + C$

**Example 1** (§8.5.67)**.** *Find the area of the region bounded by the curve* $y = \frac{10}{x^2-2x-24}$*, the* $x$*-axis, and the lines* $x = -2$ *and* $x = 2$*.*

Here is a table of all convergence tests:

**Table 10.4 Special Series and Convergence Tests**

| Series or Test | Form of Series | Condition for Convergence | Condition for Divergence | Comments |
|---|---|---|---|---|
| Geometric series | $\sum_{k=0}^{\infty} ar^k,\ a \neq 0$ | $\lvert r\rvert < 1$ | $\lvert r\rvert \geq 1$ | If $\lvert r\rvert < 1$, then $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. |
| Divergence Test | $\sum_{k=1}^{\infty} a_k$ | Does not apply | $\lim_{k\to\infty} a_k \neq 0$ | Cannot be used to prove convergence |
| Integral Test | $\sum_{k=1}^{\infty} a_k$, where $a_k = f(k)$ and $f$ is continuous, positive, and decreasing | $\int_1^{\infty} f(x)\,dx$ converges. | $\int_1^{\infty} f(x)\,dx$ diverges. | The value of the integral is not the value of the series. |
| $p$-series | $\sum_{k=1}^{\infty} \frac{1}{k^p}$ | $p > 1$ | $p \leq 1$ | Useful for comparison tests |
| Ratio Test | $\sum_{k=1}^{\infty} a_k$ | $\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert < 1$ | $\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert > 1$ | Inconclusive if $\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert = 1$ |
| Root Test | $\sum_{k=1}^{\infty} a_k$ | $\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} < 1$ | $\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} > 1$ | Inconclusive if $\lim_{k\to\infty} \sqrt[k]{\lvert a_k\rvert} = 1$ |
| Comparison Test | $\sum_{k=1}^{\infty} a_k$, where $a_k > 0$ | $a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges. | $b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges. | $\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$. |
| Limit Comparison Test | $\sum_{k=1}^{\infty} a_k$, where $a_k > 0, b_k > 0$ | $0 \leq \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges. | $\lim_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges. | $\sum_{k=1}^{\infty} a_k$ is given; you supply $\sum_{k=1}^{\infty} b_k$. |
| Alternating Series Test | $\sum_{k=1}^{\infty} (-1)^k a_k$, where $a_k > 0$ | $\lim_{k\to\infty} a_k = 0$ and $0 < a_{k+1} \leq a_k$ | $\lim_{k\to\infty} a_k \neq 0$ | Remainder $R_n$ satisfies $\lvert R_n\rvert \leq a_{n+1}$ |
| Absolute Convergence | $\sum_{k=1}^{\infty} a_k$, $a_k$ arbitrary | $\sum_{k=1}^{\infty} \lvert a_k\rvert$ converges. | | Applies to arbitrary series |

**Example 2** (Midterm 2 #8f, #8h)**.** *Do the following converge or diverge?*

1. $\sum_{k=3}^{\infty} \frac{2}{k(2-k)}$

2. $\left\{ \frac{(3k^2+2k+1)\sin(k)}{4k^3+k} \right\}_{k=1}^{\infty}$

Here is a list of commonly used Taylor series:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \text{ for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 + \cdots + (-1)^k x^k + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k, \text{ for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \quad \text{for } -1 < x \le 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k}, \quad \text{for } -1 \le x < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad \text{for } |x| \le 1$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \text{ for } |x| < \infty$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \text{ for } |x| < \infty$$

$$(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k, \text{ for } |x| < 1 \text{ and } \binom{p}{k} = \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}, \ \binom{p}{0} = 1$$

**Example 3** (§11.4 Ex. 33)**.** *Consider the differential equation $y'(t) - y = 0$ with initial condition $y(0) = 2$. Find a power series for the solution of the differential equation and identify the function represented by the power series.*