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What is on today

1 Introduction

- Please refer to the syllabus for all course details.
- Notes will be available on the course webpage (http://math.bu.edu/people/jbala/242.html) for use in each class.
- Lectures will be recorded and available on our course's Zulip.
- The first homework assignment has been posted and is due next Tuesday (February 2nd) at 12 PM EST on Gradescope. Did everyone get an invite (or two) to Gradescope?
- Please fill out the (quick!) start-of-semester survey.

2 Systems of linear equations

Lay–Lay–McDonald §1.1 pp. 2 – 12

Systems of linear equations are central to the study of linear algebra. A number of real-world scenarios can be modeled by a collection of linear equations in a large number of variables. Throughout this course, we will study some of these applications, drawing from physics, applied mathematics, economics, and engineering. Here are two examples:

- *Linear programming and optimization*: Many important management decisions in companies are made on the basis of linear programming models that use hundreds of variables. The airline industry, for instance, uses linear programs that schedule flight crews, monitor the location of aircraft, or plan scheduling of support services such as maintenance and terminal operations.
- *Electrical networks and hardware*: Engineers use simulation software to design electrical circuits and microchips involving millions of transistors. Such software relies on systems of linear equations and other tools from linear algebra.

Jumping ahead, there are many famous applications of linear algebra more broadly. For instance, Google's famous PageRank algorithm is an application of *eigenvectors*, a key topic we'll cover later in this course.

Let's start by establishing some definitions.

A *linear equation* in the variables $x_1, \ldots, x_n$ is an equation that can be written as

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where b and the coefficients $a_1, \ldots, a_n$ are real or complex numbers. The subscript n may be any positive integer. (In this course, n will typically be between 2 and 5, but in real-life, n may be very large!)

A *system of linear equations* (or a *linear system*) is a collection of one or more linear equations in the same variables $x_1, \ldots x_n$. Here's an example:

$$\left.\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3. \end{aligned}\right\} \Rightarrow ?? \quad \begin{aligned} x_1 &= \cdots \\ x_2 &= \cdots \end{aligned} \quad \text{OR not} \ldots ?$$

A *solution* of the system is a list $(s_1, \ldots, s_n)$ of numbers that makes equation a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots x_n$, respectively. In the example above, $(3, 2)$ is a solution of the linear system (in fact, it is the only solution).

The set of all possible solutions is called the *solution set* of the linear system. Two linear systems are said to be *equivalent* if they have the same solution set.

We'll say more about this in the next section, but one way to understand a system of linear equations is in terms of its solutions: it either has

1. no solution, or ⇒ inconsistent
2. exactly one solution, or
3. infinitely many solutions. ⇒ consistent

We say that a system of linear equations is *consistent* if it has either one solution or infinitely many solutions and *inconsistent* otherwise.

We introduce matrix notation to help denote linear systems. Let's look at the following linear system:

$$\begin{aligned} 1x_1 - 2x_2 + 1x_3 &= 0 \\ 0x_1 + 2x_2 - 8x_3 &= 8 \\ 5x_1 + 0x_2 - 5x_3 &= 10. \end{aligned} \tag{1}$$

The matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix} \quad \text{etc.}$$

is the *coefficient* matrix of (1) and

$$\left[\begin{array}{ccc|c} \underbrace{\begin{matrix}1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5\end{matrix}}_{\text{coeff. matrix}} & \begin{matrix}0 \\ 8 \\ 10\end{matrix}\end{array}\right] \tag{2}$$

is the *augmented matrix* of the system. The *size* of a matrix refers to how many rows and columns it has. For instance, the matrix (2) is a 3×4 matrix because it has 3 rows and 4 columns. In general, a matrix with m rows and n columns is said to be an $m \times n$ matrix.

Our goal today is to describe an algorithm for solving linear systems. The basic method is to replace one system with an equivalent system that is easier to solve. We have three operations to simplify linear systems:

Elementary row operations

1. Replace one row by the sum of itself and a multiple of another row.
2. Interchange two rows.
3. Multiply all entries in a row by a nonzero constant.

Let's see this in action:

Example 1 (1.1.11) *Solve*

$$\begin{aligned} 0x_1 + x_2 + 4x_3 &= -5 \\ x_1 + 3x_2 + 5x_3 &= -2 \\ 3x_1 + 7x_2 + 7x_3 &= 6. \end{aligned}$$

want 1 here

$$\begin{pmatrix} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{pmatrix} \xrightarrow{\text{swap}} \begin{pmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 3 & 7 & 7 & 6 \end{pmatrix} \xrightarrow{\#1(-3)+\#3} \begin{pmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & 12 \end{pmatrix}$$

take 1st row, multiply by -3, add to third row → put in third row.

wanted 0s here under the 1.

take $2 \cdot \#2 + \#3$

$$\begin{array}{c} \begin{pmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{pmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \end{array}$$

$$\cancel{0x_1 + 0x_2 + 0x_3 = 2}$$

$$\cancel{0 = 2}$$

no solutions

(inconsistent)

Here are two fundamental questions about a linear system:

1. Is the system consistent; that is, does at least one solution exist?
2. If a solution exists, is it the only one; that is, is the solution unique?

Example 2 (1.1.15) *Determine if the following system is consistent:*

$$\begin{aligned} x_1 + 3x_3 &= 2 \quad \to x_1 + 0x_2 + 3x_3 + 0x_4 = 2 \\ x_2 - 3x_4 &= 3 \quad \to 0x_1 + 1x_2 + 0x_3 - 3x_4 = 3 \\ -2x_2 + 3x_3 + 2x_4 &= 1 \quad \to 0x_1 - 2x_2 + 3x_3 + 2x_4 = 1 \\ 3x_1 + 7x_4 &= -5. \quad \to 3x_1 + 0x_2 + 0x_3 + 7x_4 = -5 \end{aligned}$$

$+0x_2$ $+0x_4$

$$\begin{pmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{pmatrix} \xrightarrow{-3\cdot\#1+\#4} \begin{pmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 0 & 0 & -9 & 7 & -11 \end{pmatrix}$$

← want this to be $0 \; 0 \cdots$

$$\xrightarrow{2\cdot\#2+\#3} \begin{pmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & -9 & 7 & -11 \end{pmatrix} \xrightarrow{3\cdot\#3+\#4} \begin{pmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & 0 & -5 & 10 \end{pmatrix}$$

→ $-5x_4 = 10$
$x_4 = -2$

$$\xrightarrow{\#4 \div (-5)} \begin{pmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

$\Rightarrow x_4 = -2$

$\Rightarrow$ there is a solution

$(x_1, x_2, x_3, x_4) = (\; , \; , \; , -2)$

when is there not a solution:

$\begin{pmatrix} 0 & \cdots & 0 & 3 \end{pmatrix}$

← or some nonzero #

$0 \neq 3$

if you end up with $\begin{pmatrix} 0 & \cdots & 0 & 1 & m \end{pmatrix}$ (or equivalent)

yes, solution

system is consistent

Example 3 (1.1.20) *Determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.*

$$\begin{array}{ccc} x_1 & x_2 & \\ \end{array}$$

$$\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix}$$

$$\begin{pmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{pmatrix} \to \boxed{\begin{pmatrix} 1 & h & -3 \\ 0 & 2h+4 & 0 \end{pmatrix}}$$

Case 1

$$\xrightarrow[\substack{\div 2h+4 \\ h \neq -2}]{} \boxed{\begin{pmatrix} 1 & h & -3 \\ 0 & \boxed{1} & 0 \end{pmatrix}}$$

$x_2 = 0$

has solution

$0 \cdot x_1 + 1 \cdot x_2 = 0 \Rightarrow x_2 = 0$

Case 2

$$\xrightarrow{h=-2} \begin{pmatrix} 1 & -2 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 - 2x_2 = -3$

has solution

all values of h !

Example 4 (1.1.18) *Do the three planes $x_1 + 2x_2 + x_3 = 4, x_2 - x_3 = 1,$ and $x_1 + 3x_2 = 0$ have at least one common point of intersection?*

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \end{pmatrix} \xrightarrow{\text{after several operations}} \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

swap

no intersection

(inconsistent)

$$\downarrow$$

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

$$\downarrow -1 \cdot \#1 + \#2$$

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & -4 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

$$\downarrow -1 \cdot \#2 + \#3$$

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

or last row $(0 \ 0 \ 0 \ -5)$

the point is $(0 \ 0 \ 0 \ \text{nonzero number})$