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What is on today

1 Row reduction and echelon forms

Lay–Lay–McDonald §1.2 pp. 12 – 24

Today we will refine the method introduced in the previous class into a row reduction *algorithm* that will allow us to analyze any system of linear equations. In the definitions that follow, a *nonzero row* or *nonzero column* in a matrix means a row or column that has at least one nonzero entry. A *leading entry* of a row refers to the leftmost nonzero entry in a nonzero row.

A rectangular matrix is in *echelon form* (or *row echelon form*) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in *reduced echelon form* or *reduced row echelon form*:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

An echelon matrix (respectively, reduced echelon matrix) is one that is in echelon form (respectively, reduced echelon form).

Example 1 *The following matrix is in echelon form. The leading entries ($\square$) may have any nonzero value. The starred entries (*) may have any value (including zero):*

$$\begin{bmatrix} \square & * & * & * \\ 0 & \square & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Property 2 above says that the leading entries form an *echelon* ("steplike") pattern that moves down and to the right through the matrix. Property 3 is a consequence of Property 2, but we include it for emphasis.

Example 2 *The following matrix is in reduced echelon form because the leading entries are 1s, and there are 0s below and above each leading 1:*

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Any nonzero matrix may be row reduced (transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form is unique:

Theorem 3 *Each matrix is row equivalent to one and only one reduced echelon matrix.*

When row operations produce an echelon form, further row operations to obtain the reduced row echelon form do not change the positions of the leading entries. Since the reduced echelon form is unique, *the leading entries are always in the same position in any echelon form obtained from a given matrix*! These leading entries correspond to leading 1s in the reduced echelon form.

A *pivot position* in a matrix A is a location that corresponds to a leading 1 in the reduced echelon form of A. A *pivot column* is a column that contains a pivot position.

A *pivot* is a nonzero number in a pivot position that is used as needed to create zeros via row operations.

Now we give the row reduction algorithm and use it in some examples:

Row Reduction Algorithm

To produce a matrix in echelon form, do the following:

1. Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.
2. Select a nonzero entry in the pivot column as a pivot. If necessary, swap rows to move this entry into the pivot position.
3. Use row replacement operations to create zeros in all positions below the pivot.
4. Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply Steps 1-3 above to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

If we want the reduced echelon form, we carry out one more step:

5. Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by rescaling the row.

Example 4 (1.2.3) *Row reduce the matrix below to reduced echelon form and locate its pivot columns.*

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}.$$

Example 5 (1.2.4) *Apply elementary row operations to transform the following matrix into reduced echelon form:*

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}.$$

The row reduction algorithm leads directly to a description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system. Suppose that the augmented matrix of a linear system has been changed into the equivalent reduced echelon form

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are three variables because the augmented matrix has four columns. The associated system of equations is

$$\begin{aligned} x_1 \qquad - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0. \end{aligned} \tag{1}$$

The variables x_1 and x_2 corresponding to pivot columns are called basic variables. The other variable x_3 is called a free variable.

Whenever a system is consistent, as in (1), the solution set can be described explicitly by solving the reduced system of equations for the basic variables in terms of the free variables. So for instance, the above would give

$$\begin{aligned} &x_1 = 1 + 5x_3 \\ &x_2 = 4 - x_3 \\ &x_3 \text{ is free.} \end{aligned}$$

Example 6 (1.2.7) *Find the general solution of the linear system whose augmented matrix is the following:*

$$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}.$$

Although a nonreduced echelon form is a poor tool for solving a system, this form is just the right device for answering two fundamental question (existence and uniqueness) posed in the previous class.

Example 7 *Determine the existence and uniqueness of the solutions to the system*

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15. \end{aligned}$$

Theorem 8 (Existence and uniqueness) *A linear system is consistent if and only if the rightmost column of the augmented matrix is* not *a pivot column—that is, if and only if an echelon form of the augmented matrix has no row of the form*

$$[0 \ \cdots \ 0 \quad b],$$

with b nonzero. If a linear system is consistent, then the solution set contains either a unique solution (no free variables) or infinitely many solutions (at least one free variable).

The following summarizes how to find and describe all solutions of a linear system:

Using row reduction to solve a linear system

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix in Step 3.
5. Rewrite each nonzero equation from Step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.