Professor Jennifer Balakrishnan, *jbala@bu.edu*

What is on today

1 Vector equations

Lay–Lay–McDonald §1.3 pp. 24 – 35

Many key ideas in the study of linear systems can be described in terms of *vectors* and *vector spaces*. This will be made more precise later in the course. Today we'll show how vectors can be used to rewrite systems of linear equations; to do this, we'll take our working definition of vector as "an ordered list of numbers."

A *column vector*, or simply a *vector*, is a matrix with only one column. For example,

$$\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 3 \\ 0 \\ -2.1 \\ \pi \end{bmatrix}$$

are both vectors.

We say that two vectors are equal if and only if their corresponding entries are equal. So, for instance,

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} \neq \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

are not equal, since the ordering of the numbers matters.

Given two vectors $\mathbf{u}, \mathbf{v}$, their sum is the vector $\mathbf{u} + \mathbf{v}$ obtained by adding corresponding entries. The scalar multiple of $\mathbf{u}$ by c is the vector $c \cdot \mathbf{u}$ obtained by multiplying each entry in $\mathbf{u}$ by c.

Example 1. *Let*

$$\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

Compute $4\mathbf{u}, -3\mathbf{v}, 4\mathbf{u} - 3\mathbf{v}$.

We can think of vectors geometrically. For instance, the column vector

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

can be identified with the point $(a, b) \in \mathbb{R}^2$ in the coordinate plane. The sum of two vectors has a useful geometric description:

Theorem 2 (Parallelogram rule for addition). *If* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ *are represented as points in the plane, then* $\mathbf{u} + \mathbf{v}$ *corresponds to the fourth vertex of the parallelogram whose other vertices are* $\mathbf{u}, \mathbf{0}$, *and* $\mathbf{v}$.

If n is a positive integer, $\mathbb{R}^n$ denotes the collection of all ordered n-tuples of n real numbers, usually written as $n \times 1$ column matrices, such as

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

The vector whose entries are all zero is called the *zero vector* and is denoted by $\mathbf{0}$. Equality of vectors in $\mathbb{R}^n$ and the operations of scalar multiplication and vector addition are defined entry-by-entry. These operations on vectors have the following properties:

Algebraic properties of $\mathbb{R}^n$
For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and scalars c, d:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$, where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$

Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$ and scalars $c_1, \ldots, c_p$, the vector $\mathbf{y}$ described by

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

is called a *linear combination* of $v_1, \ldots v_p$ with weights $c_1, \ldots c_p$. The next example connects a problem about linear combinations of vectors to the fundamental existence question we've been looking at:

Example 3. *Let*

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$$

Determine whether $\mathbf{b}$ *can be written as a linear combination of* $\mathbf{a}_1, \mathbf{a}_2$.

A vector equation

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$[\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}]. \tag{1}$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (1).

One of the key ideas in linear algebra is to study the set of all vectors that can be generated or written as a linear combination of a fixed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of vectors.

Definition 4. *If* $\mathbf{v}_1, \ldots \mathbf{v}_p \in \mathbb{R}^n$ *then the set of all linear combinations of* $\mathbf{v}_1, \ldots, \mathbf{v}_p$ *is denoted by Span* $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ *and is called the subset of* $\mathbb{R}^n$ *spanned (or generated) by* $\mathbf{v}_1, \ldots, \mathbf{v}_p$. *That is, Span* $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ *is the collection of all vectors that can be written in the form*

$$c_1\mathbf{v}_1 + \cdots c_p\mathbf{v}_p$$

with $c_1, \ldots c_p$ *scalars.*

We can also consider a geometric description of Span $\{\mathbf{u}, \mathbf{v}\}$, as in the following example.

Example 5. *Let*

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}.$$

Then Span $\{\mathbf{a}_1, \mathbf{a}_2\}$ *is a plane through the origin in* $\mathbb{R}^3$. *Is* $\mathbf{b}$ *in that plane?*

Now we apply our study of linear combinations to an application:

Example 6 (1.3.27)**.** *A mining company has two mines. One day's operation at mine #1 produces ore that contains 20 metric tons of copper and 550 kilograms of silver, while one day's operation at mine #2 produces ore that contains 30 metric tons of copper and 500 kilograms of silver. Let* $\mathbf{v}_1 = \begin{bmatrix} 20 \\ 550 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} 30 \\ 500 \end{bmatrix}$. *Then* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *represent the "output per day" of mine #1 and mine #2, respectively.*

1. *What physical interpretation can be given to the vector* $5\mathbf{v}_1$*?*
2. *Suppose the company operates mine #1 for* x_1 *days and mine #2 for* x_2 *days. Write a vector equation whose solution gives the number of days each mine should operate in order to produce 150 metric tons of copper and 2825 kilograms of silver.*

2 The matrix equation $Ax = b$

Lay–Lay–McDonald §1.4 pp. 35 – 36

A fundamental idea in linear algebra is to view a linear combination of vectors as the product of a matrix and a vector. The following allows us to rephrase some of the ideas in the previous class in new ways:

If A is an $m \times n$ matrix with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$ and if $\mathbf{x} \in \mathbb{R}^n$, then the product of A and $\mathbf{x}$, denoted by $A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights; that is:

$$A\mathbf{x} = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n.$$

Example 7. *For $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^m$, write the linear combination $3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3$ as a matrix times a vector.*

Previously, we looked at writing a system of linear equations as a vector equation involving a linear combination of vectors. For example, the system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned}$$

is equivalent to

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

As in the previous example, the linear combination on the left side is a matrix times a vector, and we may rewrite the equation as

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}. \tag{2}$$

Equation (2) has the form $A\mathbf{x} = \mathbf{b}$. Such an equation is called a matrix equation. It turns out that any system of linear equations can be written as an equivalent matrix equation. We will use this observation throughout the course. Here is the formal result:

Theorem 8. *If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots \mathbf{a}_n$ and if $\mathbf{b} \in \mathbb{R}^m$, the matrix equation*

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which in turn has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \mathbf{a}_n \quad \mathbf{b}].$$