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What is on today

1 Introduction to linear transformations

Lay–Lay–McDonald §1.8 pp. 63 – 69

Now we'll look at transforming vectors under matrix multiplication, which introduces the idea of *linear transformations*. For example, in the equation

$$\begin{array}{ccc} A & \mathbf{x} & = \mathbf{b} \\ \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} & = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \end{array}$$

multiplication by the matrix A transforms $\mathbf{x}$ into $\mathbf{b}$, and in the equation

$$\begin{array}{ccc} A & \mathbf{u} & = \mathbf{0} \\ \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix} & = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}$$

multiplication by A transforms $\mathbf{u}$ into $\mathbf{0}$.

From this new point of view, solving the equation $A\mathbf{x} = \mathbf{b}$ amounts to finding all vectors $\mathbf{x}$ in $\mathbb{R}^4$ that are transformed into the vector $\mathbf{b}$ in $\mathbb{R}^2$ under multiplication by A. Here we introduce some new terminology to further this viewpoint.

A *transformation* (or *function* or *mapping*) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector in $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the *domain* of T and $\mathbb{R}^m$ is called the *codomain* of T. The notation

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$. For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the *image* of $\mathbf{x}$. The set of all images $T(\mathbf{x})$ is called the *range* of T.

Example 1. *Let* $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ *and define a transformation* $T : \mathbb{R}^2 \to \mathbb{R}^3$ *by* $T(\mathbf{x}) = A\mathbf{x}$, *so that*

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}.$$

v_1 v_2

$\{x_1v_1 + x_2v_2\} = \text{Span}\{v_1, v_2\}$

$Ax = b$

$\underbrace{m \times n}_{\text{matrix}}$ $\underbrace{n \times 1}_{\text{4c it's a vector}}$

x vector

b vector

$\Rightarrow b = m \times 1$

range $= \text{Span}\{v_1, v_2\}$ $\mathbb{R}^n \to \mathbb{R}^m$.

1. *Find* $T(\mathbf{u})$, *the image of* $\mathbf{u}$ *under the transformation* T.

$$T(u) = A \cdot u = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1\cdot 2 + 3 \\ 3 \cdot 2 + 5(-1) \\ -1 \cdot 2 + 7(-1) \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -9 \end{pmatrix}$$

2. *Find an* $\mathbf{x}$ *in* $\mathbb{R}^2$ *whose image under* T *is* $\mathbf{b}$.

want: $T(x) = b$, $T(x) = Ax$ → solve $Ax = b$ for x

$$\begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix} x = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 3 \\ -1 & 7 & -5 \\ 3 & 5 & 2 \end{pmatrix} \overset{\#1+\#2}{\sim} \begin{pmatrix} 1 & -3 & 3 \\ 0 & 4 & -2 \\ 3 & 5 & 2 \end{pmatrix} \overset{-3\cdot\#1+\#3}{\sim} \begin{pmatrix} 1 & -3 & 3 \\ 0 & 4 & -2 \\ 0 & 14 & -7 \end{pmatrix}$$

rowreduce

$$\overset{\frac{1}{4}\cdot\#2}{\Rightarrow} \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 14 & -7 \end{pmatrix} \overset{-14\cdot\#2+\#3}{\sim} \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \overset{3\cdot\#2+\#1}{\Rightarrow} \begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

x_1 x_2

$x_1 = 3/2$

$x_2 = -1/2$

$$\Rightarrow x = \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix}$$

3. *Is there more than one* $\mathbf{x}$ *whose image under* T *is* $\mathbf{b}$*?*

no - there is just one x, from the work in 2 (there are no free variables)

4. *Determine if* $\mathbf{c}$ *is in the range of the transformation* T.

want x s.t.: $Ax = c$

$$\Rightarrow \begin{pmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 3 \\ -1 & 7 & 5 \\ 3 & 5 & 2 \end{pmatrix} \overset{\#1+\#2,\ -3\cdot\#1+\#3}{\sim} \begin{pmatrix} 1 & -3 & 3 \\ 0 & 4 & 8 \\ 0 & 14 & -7 \end{pmatrix}$$

$$\overset{\frac{1}{4}\cdot\#2,\ \frac{1}{7}\cdot\#3}{\Rightarrow} \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & -1 \end{pmatrix} \overset{-2\cdot\#2+\#3}{\sim} \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{pmatrix}$$

x_1 x_2

this means $0 = -5$, which is a contradiction $\Rightarrow$ no, c is not in the range of the transformation T

The next two matrix transformations each have a nice geometric interpretation.

Example 2. *If* $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, *then the transformation* $\mathbf{x} \mapsto A\mathbf{x}$ *projects points in* $\mathbb{R}^3$ *into the* x_1x_2*-plane, because*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}.$$

Example 3. *Let* $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. *The transformation* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by* $T(\mathbf{x}) = A(\mathbf{x})$ *is called a* shear transformation. *This transformation sends a square to a parallelogram, deforming the square as if the top of the square were pushed to the right while the base is held fixed.*

Recall that we saw earlier that the transformation $\mathbf{x} \mapsto A\mathbf{x}$ has the properties

$$A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}, \quad A(c\mathbf{u}) = cA\mathbf{u},$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all scalars c. These key properties lead us to the formal definition of a linear transformation.

Definition 4. *A transformation T is* linear *if*

1. *$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T,*
2. *$T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.*

Remark 5. *Note that every matrix transformation is a linear transformation.*

Here are a few more useful facts, both of which can be derived from the above. If T is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$ and $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$ for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T and all scalars c, d.

Example 6. *Given a scalar r, define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = r\mathbf{x}$. T is called a contraction when $0 \le r \le 1$ and a dilation when $r > 1$. Let $r = 2$ and show that T is a linear transformation.*

Let $T(x) = 2x$. We have $T(u+v) = 2(u+v) = 2u + 2v = T(u) + T(v)$ ✓

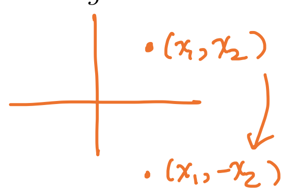
We also have $T(cu) = 2 \cdot cu = c \cdot 2u = c \cdot T(u)$ ✓

So we conclude that T is linear.

Example 7. *Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Give a geometric description of the transformation $\mathbf{x} \mapsto A\mathbf{x}$.*

$$\overset{A}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \overset{x}{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$$

$-x_2 = 0 \cdot x_1 + (-1) \cdot x_2$

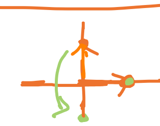


reflection over x-axis.

Another perspective:

T sends $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

T sends $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ -1 \end{pmatrix}$



Example 8 (1.8.30)**.** *An affine transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ has the form $T(x) = A\mathbf{x} + \mathbf{b}$, with A an $m \times n$ matrix and $\mathbf{b}$ in $\mathbb{R}^m$. Show that T is not a linear transformation when $\mathbf{b} \neq \mathbf{0}$. (Affine transformations are important in compute graphics.)*

If T were a linear transformation: $T(u+v) = T(u) + T(v)$

$A(u+v) + b = Au + b + Av + b$

$Au + Av + b = Au + Av + 2b$

$\Rightarrow b = 0 \Rightarrow$ If $b \neq 0$, then $T(u+v) \neq T(u) + T(v)$, hence not linear.

2 The matrix of a linear transformation

Lay–Lay–McDonald §1.9 pp. 71 – 78

Whenever a linear transformation T arises geometrically, it's an interesting problem to compute the corresponding matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$. (Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is actually a matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$.) The key to finding A is to observe that T is completely determined by what it does to the columns of the $n \times n$ identity matrix I_n.

Example 9. *The columns of* $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *are* $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and* $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. *Suppose T is a linear transformation from* $\mathbb{R}^2$ *to* $\mathbb{R}^3$ *such that* $T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$ *and* $T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$. *Find a formula for the image of an arbitrary* $\mathbf{x}$ *in* $\mathbb{R}^2$.

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is an arbitrary elt. of $\mathbb{R}^2$

$= x_1 e_1 + x_2 e_2$

$= x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow T(x) = T(x_1 e_1 + x_2 e_2)$

$= x_1 T(e_1) + x_2 T(e_2)$

$= x_1 \begin{pmatrix} 5 \\ -7 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 8 \\ 0 \end{pmatrix} = \begin{pmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 \end{pmatrix}$

Theorem 10. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that*

$$T(\mathbf{x}) = A\mathbf{x} \quad \textit{for all } \mathbf{x} \in \mathbb{R}^n.$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$:

$$A = [T(\mathbf{e}_1) \quad \cdots \quad T(\mathbf{e}_n)]. \tag{1}$$

$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

The matrix A in (1) is called the *standard matrix for the linear transformation T*.

$e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

etc.

Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be viewed as a matrix transformation, and vice versa!

We practice with finding the standard matrix for linear transformations in the next two examples:

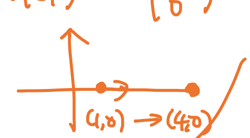
Example 11. *Find the standard matrix A for the dilation $T(\mathbf{x}) = 4\mathbf{x}$ for $\mathbf{x}$ in $\mathbb{R}^2$.*

$A = [T(e_1) \quad T(e_2)]$

$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

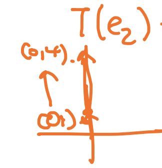
$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$T(e_1) = 4\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$



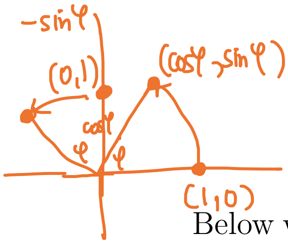
$e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$T(e_2) = 4\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$



$T(x) = 4x$

Example 12. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle φ, with counterclockwise rotation for a positive angle. Such a transformation is linear. Find the standard matrix A of this transformation.*



$A = [\,T(e_1)\ \ T(e_2)\,]$ $\qquad e_1 = \begin{pmatrix}1\\0\end{pmatrix} \qquad e_2 = \begin{pmatrix}0\\1\end{pmatrix}$

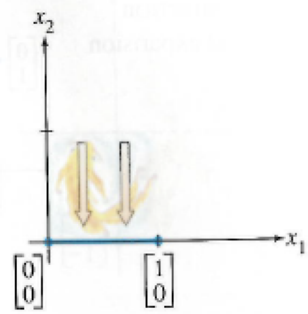
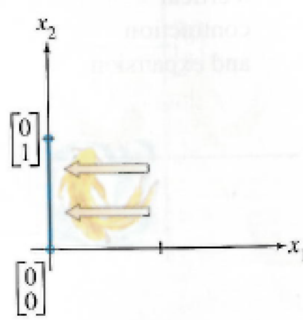
$$\begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix}$$

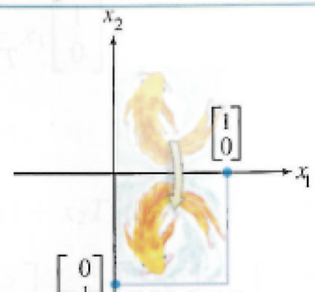
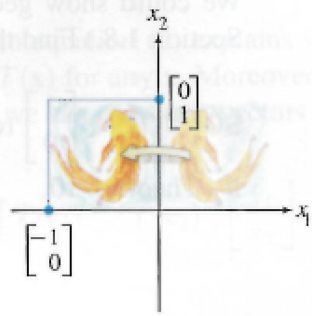
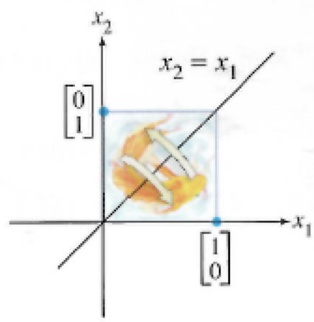
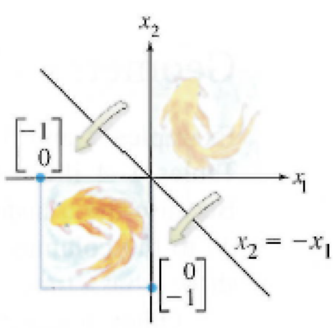
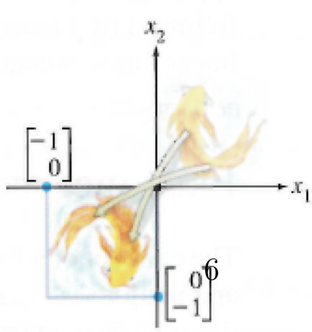
ϕ φ

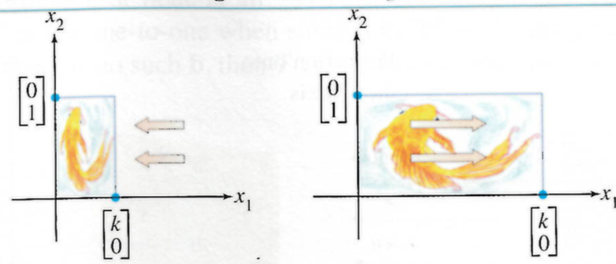
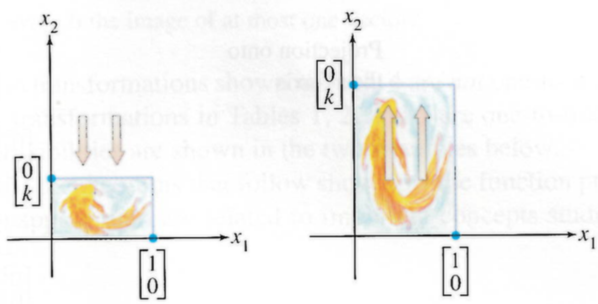
\phi \varphi

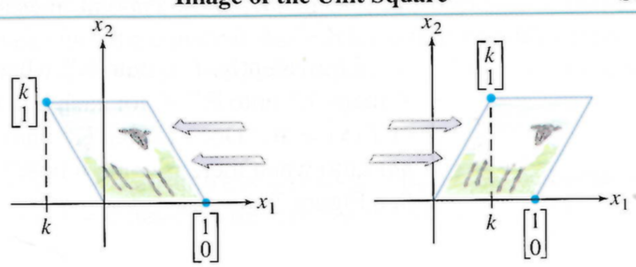
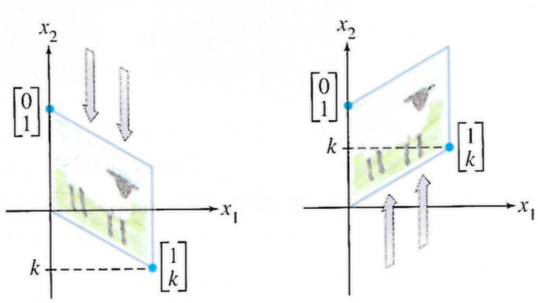
"fee"

Below we reproduce some helpful figures from the textbook (§1.8, Tables 1–4) illustrating various geometric linear transformations (projections, reflections, contractions and cxpansions, and shears, respectively) of $\mathbb{R}^2$.

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1-axis		$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$
Projection onto the x_2-axis		$\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1-axis		$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection through the x_2-axis		$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection through the line $x_2 = x_1$		$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Reflection through the line $x_2 = -x_1$		$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
Reflection through the origin		$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Transformation	Image of the Unit Square	Standard Matrix
Horizontal contraction and expansion	 $0 < k < 1$ $k > 1$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Vertical contraction and expansion	 $0 < k < 1$ $k > 1$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

Transformation	Image of the Unit Square	Standard Matrix
Horizontal shear	 $k < 0$ $k > 0$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear	 $k < 0$ $k > 0$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$