What is on today

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1 Introduction to linear transformations

Lay–Lay–McDonald §1.8 pp. 63 – 69

Now we’ll look at transforming vectors under matrix multiplication, which introduces the idea of linear transformations. For example, in the equation

\[
A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = b
\]

\[
\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}
\]

multiplication by the matrix \(A\) transforms \(x\) into \(b\), and in the equation

\[
A \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0
\]

\[
\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

multiplication by \(A\) transforms \(u\) into \(0\).

From this new point of view, solving the equation \(Ax = b\) amounts to finding all vectors \(x\) in \(\mathbb{R}^4\) that are transformed into the vector \(b\) in \(\mathbb{R}^2\) under multiplication by \(A\). Here we introduce some new terminology to further this viewpoint.

A transformation (or function or mapping) \(T\) from \(\mathbb{R}^n\) to \(\mathbb{R}^m\) is a rule that assigns to each vector in \(x\) in \(\mathbb{R}^n\) a vector \(T(x)\) in \(\mathbb{R}^m\). The set \(\mathbb{R}^n\) is called the domain of \(T\) and \(\mathbb{R}^m\) is called the codomain of \(T\). The notation

\[
T : \mathbb{R}^n \rightarrow \mathbb{R}^m
\]

indicates that the domain of \(T\) is \(\mathbb{R}^n\) and the codomain is \(\mathbb{R}^m\). For \(x\) in \(\mathbb{R}^n\), the vector \(T(x)\) in \(\mathbb{R}^m\) is called the image of \(x\). The set of all images \(T(x)\) is called the range of \(T\).
Example 1. Let \( A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}, c = \begin{bmatrix} 3 \\ 2 \end{bmatrix}\) and define a transformation \( T : \mathbb{R}^2 \to \mathbb{R}^3 \) by \( T(x) = Ax \), so that

\[
T(x) = Ax = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}.
\]

1. Find \( T(u) \), the image of \( u \) under the transformation \( T \).

\[
T(u) = A \cdot u = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 2 + (-3) \cdot (-1) \\ 3 \cdot 2 + 5 \cdot (-1) \\ -1 \cdot 2 + 7 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 5 \\ -9 \end{pmatrix}.
\]

2. Find an \( x \) in \( \mathbb{R}^2 \) whose image under \( T \) is \( b \).

\[
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

\[
\text{want: } T(x) = b \quad \Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}
\]

3. Is there more than one \( x \) whose image under \( T \) is \( b \)?

No - there is just one \( x \), from the work in 2 (there are no free variables).

4. Determine if \( c \) is in the range of the transformation \( T \).

\[
\text{want: } Ax = c \quad \Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{which means }\quad x_1 = 3/2, x_2 = -1/2.
\]

The next two matrix transformations each have a nice geometric interpretation.

Example 2. If \( A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \), then the transformation \( x \mapsto Ax \) projects points in \( \mathbb{R}^3 \) into the \( x_1x_2 \)-plane, because

\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}.
\]

Example 3. Let \( A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \). The transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( T(x) = A(x) \) is called a shear transformation. This transformation sends a square to a parallelogram, deforming the square as if the top of the square were pushed to the right while the base is held fixed.
Recall that we saw earlier that the transformation $\mathbf{x} \mapsto A\mathbf{x}$ has the properties

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}, \quad A(c\mathbf{u}) = cA\mathbf{u},$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all scalars $c$. These key properties lead us to the formal definition of a linear transformation.

**Definition 4.** A transformation $T$ is linear if

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of $T$,
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars $c$ and all $\mathbf{u}$ in the domain of $T$.

**Remark 5.** Note that every matrix transformation is a linear transformation.

Here are a few more useful facts, both of which can be derived from the above. If $T$ is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$ and $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$ for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of $T$ and all scalars $c, d$.

**Example 6.** Given a scalar $r$, define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = r\mathbf{x}$. $T$ is called a contraction when $0 \leq r \leq 1$ and a dilation when $r > 1$. Let $r = 2$ and show that $T$ is a linear transformation.

Let $T(\mathbf{x}) = 2\mathbf{x}$. We have $T(\mathbf{u} + \mathbf{v}) = 2(\mathbf{u} + \mathbf{v}) = 2\mathbf{u} + 2\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$. We also have $T(c\mathbf{u}) = 2c\mathbf{u} = c(2\mathbf{u}) = cT(\mathbf{u})$. So we conclude that $T$ is linear.

**Example 7.** Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Give a geometric description of the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

This represents reflection over the $x$-axis.

**Example 8** (1.8.30). An affine transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ has the form $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, with $A$ an $m \times n$ matrix and $\mathbf{b}$ in $\mathbb{R}^m$. Show that $T$ is not a linear transformation when $\mathbf{b} \neq \mathbf{0}$. (Affine transformations are important in computer graphics.)

If $T$ were a linear transformation:

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

$$A(\mathbf{u} + \mathbf{v}) + \mathbf{b} = A\mathbf{u} + A\mathbf{v} + \mathbf{b}$$

$$A\mathbf{u} + A\mathbf{v} + \mathbf{b} = A\mathbf{u} + A\mathbf{v} + 2\mathbf{b}$$

$$\Rightarrow \quad c = 0 \quad \Rightarrow \quad \text{if } \mathbf{b} \neq \mathbf{0}, \text{ then } T(\mathbf{u} + \mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v}),$$

hence not linear.

2 The matrix of a linear transformation
Whenever a linear transformation \( T \) arises geometrically, it’s an interesting problem to compute the corresponding matrix transformation \( \mathbf{x} \mapsto \mathbf{A}\mathbf{x} \). (Every linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) is actually a matrix transformation \( \mathbf{x} \mapsto \mathbf{A}\mathbf{x} \).) The key to finding \( \mathbf{A} \) is to observe that \( T \) is completely determined by what it does to the columns of the \( n \times n \) identity matrix \( I_n \).

**Example 9.** The columns of \( I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) are \( \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). Suppose \( T \) is a linear transformation from \( \mathbb{R}^2 \) to \( \mathbb{R}^3 \) such that \( T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \) and \( T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} \). Find a formula for the image of an arbitrary \( \mathbf{x} \) in \( \mathbb{R}^2 \).

\[
\begin{align*}
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & \quad \text{is an arbitrary elt. of } \mathbb{R}^2 \\
\Rightarrow T(\mathbf{x}) &= T(x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2) \\
&= x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) \\
&= x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 5x_1 -3x_2 \\ -7x_1 +8x_2 \\ 2x_1 \end{bmatrix}
\end{align*}
\]

**Theorem 10.** Let \( T : \mathbb{R}^n \to \mathbb{R}^m \) be a linear transformation. Then there exists a unique matrix \( \mathbf{A} \) such that \( T(\mathbf{x}) = \mathbf{A}\mathbf{x} \) for all \( \mathbf{x} \in \mathbb{R}^n \).

In fact, \( \mathbf{A} \) is the \( m \times n \) matrix whose \( j \)th column is the vector \( T(\mathbf{e}_j) \), where \( \mathbf{e}_j \) is the \( j \)th column of the identity matrix in \( \mathbb{R}^n \):

\[
\mathbf{A} = [T(\mathbf{e}_1) \quad \cdots \quad T(\mathbf{e}_n)].
\]

(1)

The matrix \( \mathbf{A} \) in (1) is called the **standard matrix for the linear transformation** \( T \).

Every linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) can be viewed as a matrix transformation, and vice versa!

We practice with finding the standard matrix for linear transformations in the next two examples:

**Example 11.** Find the standard matrix \( \mathbf{A} \) for the dilation \( T(\mathbf{x}) = 4\mathbf{x} \) for \( \mathbf{x} \) in \( \mathbb{R}^2 \).

\[
\begin{align*}
\mathbf{A} &= \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} \\
&= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}
\end{align*}
\]
Example 12. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle $\varphi$, with counterclockwise rotation for a positive angle. Such a transformation is linear. Find the standard matrix $A$ of this transformation.

Below we reproduce some helpful figures from the textbook (§1.8, Tables 1–4) illustrating various geometric linear transformations (projections, reflections, contractions and expansions, and shears, respectively) of $\mathbb{R}^2$. 

![Transformation Diagrams]
<table>
<thead>
<tr>
<th>Transformation</th>
<th>Image of the Unit Square</th>
<th>Standard Matrix</th>
</tr>
</thead>
</table>
| Reflection through the $x_1$-axis | ![Diagram](image1)      | \[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\] |
| Reflection through the $x_2$-axis | ![Diagram](image2)      | \[
\begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}
\] |
| Reflection through the line $x_2 = x_1$ | ![Diagram](image3) | \[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\] |
| Reflection through the line $x_2 = -x_1$ | ![Diagram](image4) | \[
\begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix}
\] |
| Reflection through the origin    | ![Diagram](image5)      | \[
\begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}
\] |
### Transformation

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Image of the Unit Square</th>
<th>Standard Matrix</th>
</tr>
</thead>
</table>
| **Horizontal contraction and expansion** | ![Image](image1.png) | \[
\begin{bmatrix}
  k & 0 \\
  0 & 1 \\
\end{bmatrix}
\] |
| **Vertical contraction and expansion** | ![Image](image2.png) | ![Matrix](matrix2.png) |
| 0 < \(k\) < 1         | 0 < \(k\) < 1          | \(k > 1\)       |
| \(k > 1\)              | \(k > 1\)              |                 |

### Transformation

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Image of the Unit Square</th>
<th>Standard Matrix</th>
</tr>
</thead>
</table>
| **Horizontal shear**   | ![Image](image3.png)    | \[
\begin{bmatrix}
  1 & k \\
  0 & 1 \\
\end{bmatrix}
\] |
| **Vertical shear**     | ![Image](image4.png)    | \[
\begin{bmatrix}
  1 & 0 \\
  k & 1 \\
\end{bmatrix}
\] |
| \(k < 0\)             | \(k < 0\)             | \(k > 0\)       |
| \(k > 0\)              | \(k > 0\)              |                 |