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What is on today

1 The matrix of a linear transformation

Lay–Lay–McDonald §1.9 pp. 71 – 78

Now we translate our earlier existence and uniqueness questions about solutions of linear systems into questions about linear transformations, via the following terminology.

Definition 1. *A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be* onto $\mathbb{R}^m$ *if each $\mathbf{b} \in \mathbb{R}^m$ is the image of at least one $\mathbf{x} \in \mathbb{R}^n$.*

Equivalently, T is onto $\mathbb{R}^m$ when the range of T is all of the codomain $\mathbb{R}^m$. So the question "Does T map $\mathbb{R}^n$ onto $\mathbb{R}^m$?" is an existence question. The mapping T is not onto when there is some $\mathbf{b} \in \mathbb{R}^m$ for which the equation $T(\mathbf{x}) = \mathbf{b}$ has no solution.

Definition 2. *A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be* one-to-one *if each $\mathbf{b} \in \mathbb{R}^m$ is the image of at most one $\mathbf{x} \in \mathbb{R}^n$.*

Equivalently, T is one-to-one if, for each $\mathbf{b} \in \mathbb{R}^m$, the equation $T(\mathbf{x}) = \mathbf{b}$ has either a unique solution or none at all. "Is T one-to-one?" is a uniqueness question.

Example 3. *Let T be the linear transformation whose standard matrix is $A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$. Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?*

Theorem 4. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.*

Theorem 5. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then*

1. *T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.*
2. *T is one-to-one if and only if the columns of A are linearly independent.*

Example 6. *Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is a one-to-one linear transformation. Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?*

2 Linear models in business and science

Lay–Lay–McDonald §1.10 pp. 81 – 86

Today we'll look at mathematical models that are all linear – each describes a problem by means of a linear equation, usually in vector or matrix form. The first problem is about nutrition but is representative of a general technique in linear programming. The second model introduces the concept of a linear difference equation, which is a very useful tool for studying dynamic processes in a number of fields such as engineering, ecology, economics, telecommunications, and management.

Example 7 (1.10.1)**.** *The container of a breakfast cereal lists the number of calories and the amounts of protein, carbohydrate, and fat contained in one serving of the cereal. The amounts for two common cereals are given below. Suppose a mixture of these two cereals is to be prepared that contains exactly 295 calories, 9 g of protein, 48 g of carbohydrate, and 8 g of fat.*

Nutrient	*General Mills Cheerios®*	*Quaker® 100% Natural Cereal*
Calories	*110*	*130*
Protein (g)	*4*	*3*
Carbohydrate (g)	*20*	*18*
Fat (g)	*2*	*5*

1. *Set up a vector equation for this problem. What do the variables represent?*

2. *Write an equivalent matrix equation, and then determine if the desired mixture of the two cereals can be prepared.*

In many fields such as ecology, economics, and engineering, a need arises to mathematically model a dynamic system that changes over time. Several features of the system are each measured at discrete time intervals, producing a sequence of vectors $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$. The entries in $\mathbf{x}_k$ provide information about the state of the system at the time of the kth measurement.

If there is a matrix A such that $\mathbf{x}_1 = A\mathbf{x}_0, \mathbf{x}_2 = A\mathbf{x}_1$, and in general,

$$\mathbf{x}_{k+1} = A\mathbf{x}_k, \qquad k = 0, 1, 2, \ldots \tag{1}$$

then (1) is called a *linear difference equation* (or *recurrence relation*). Given such an equation, one can compute $\mathbf{x}_1, \mathbf{x}_2$ and so on, provided $\mathbf{x}_0$ is known.

Example 8 (1.10.9)**.** *In a certain region, about 7% of a city's population moves to the surrounding suburbs each year, and about 5% of the suburban population moves into the city. In 2015, there were 800,000 residents in the city and 500,000 in the suburbs. Set up a difference equation that describes this situation, where $\mathbf{x}_0$ is the initial population in 2015. Then estimate the populations in the city and in the suburbs 2 years later, in 2017. (Ignore other factors that might influence the population sizes.)*

3 Matrix operations

Lay–Lay–McDonald §2.1 pp. 94 – 102

In this chapter, the goal is to perform various algebraic operations with matrices. We begin by establishing some useful terminology.

If A is an $m \times n$ matrix, then the entry in the ith row and jth column of A is denoted by a_{ij}, and is called the (i, j)th entry of A. The *diagonal* entries in an $m \times n$ matrix $A = [a_{ij}]$ are $a_{11}, a_{22}, a_{33}, \ldots$, and they form the diagonal of A. A *diagonal matrix* is a square $n \times n$ matrix whose nondiagonal entries are zero. An $m \times n$ matrix whose entries are all zero is a *zero matrix* and is written as 0. The sum $A + B$ of matrices A and B has entries given by the corresponding sum of entries in A and B. The sum is defined only when A and B are the same size.

Example 9. *Let* $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}, C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$. *Can we compute* $A + B$*? What about* $A + C$*? What about* $A - 2B$*?*

Theorem 10. *Let* A, B, C *be matrices of the same size, and let* r *and* s *be scalars.*

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A$

Matrix multiplication is somewhat more subtle than matrix addition. One thing to notice is that the product AB of a matrix A of size $m \times n$ and a matrix B of size $p \times q$ is only defined if $n = p$; that is, the number of columns of A must match the number of rows of B. If $n = p$, that is, if we multiply A of size $m \times n$ and B of size $n \times q$, then the product AB has size $m \times q$. Here is the rule for computing AB:

If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i, j)th entry in AB, and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

Example 11. *Compute AB, where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$. Can we compute BA?*

Example 12. *Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$. Compute AB and BA. What do you notice?*

Here are some useful properties of matrix multiplication:

Theorem 13. *Let A be an $m \times n$ matrix and let B and C have sizes for which the indicated sums and products are defined.*

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$ *for any scalar* r
5. $I_m A = A = AI_n$

Here are some surprises:

1. In general, $AB \neq BA$.
2. The cancellation laws do not hold for matrix multiplication. That is, if $AB = AC$, then it is not true in general that $B = C$.
3. If a product AB is the zero matrix, you cannot conclude in general that either $A = 0$ or $B = 0$.

Given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

Example 14. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -3 \\ -4 & 0 & 5 \end{bmatrix}$. Compute $A^T, B^T, (A^T)^T, (B^T)^T, AB, (AB)^T$, and $B^T A^T$. What do you notice?*

Theorem 15. *Let A and B denote matrices whose sizes are appropriate for the following sums and products.*

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. *For any scalar r,* $(rA)^T = rA^T$
4. $(AB)^T = B^T A^T$