Professor Jennifer Balakrishnan, *jbala@bu.edu*

# What is on today



# 1 Characterizations of invertible matrices

Lay–Lay–McDonald §2.3 pp. 113 – 116

Recall that matrix multiplication corresponds to composition of linear transformations. When a matrix $A$ is invertible, the equation $A^{-1}A\mathbf{x} = \mathbf{x}$ can be viewed as a statement about linear transformations.

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *invertible* if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n \tag{1}$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n. \tag{2}$$

The next result shows that if such an $S$ exists, it is unique and must be a linear transformation. We call $S$ the *inverse* of $T$ and write it as $T^{-1}$.

**Theorem 1.** *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, and let $A$ be the standard matrix for $T$. Then $T$ is invertible if and only if $A$ is an invertible matrix. In that case, the linear transformation $S$ given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying (1) and (2).*

**Example 2** (2.3.33)**.** *The following transformation $T$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$:*

$$T(x_1, x_2) = (-5x_1 + 9x_2, 4x_1 - 7x_2).$$

*Show that $T$ is invertible and find a formula for $T^{-1}$.*

**Example 3.** *What can you say about a one-to-one linear transformation $T$ from $\mathbb{R}^n$ to $\mathbb{R}^n$?*

# 2 Matrix factorization

Lay–Lay–McDonald §2.5 pp. 125 – 129

A *factorization* of a matrix $A$ is an equation that expresses $A$ as a product of two or more matrices. In the language of computer science, the expression of $A$ as a product amounts to a *preprocessing* of the data in $A$, organizing that data into two or more parts whose structures are more useful in some way, perhaps more accessible for computation.

The *LU* factorization (described below) is motivated by the common real-life problem of solving a sequence of equations, all with the same coefficient matrix:

$$A\mathbf{x} = \mathbf{b}_1, \quad A\mathbf{x} = \mathbf{b}_2, \quad \ldots, \quad A\mathbf{x} = \mathbf{b}_p. \tag{3}$$

When $A$ is invertible, one could compute $A^{-1}$ and then compute $A^{-1}\mathbf{b}_1, A^{-1}\mathbf{b}_2$, and so on. However, it is more efficient to solve the first equation in sequence (3) by row reduction and obtain an LU factorization at the same time. Then the remaining equations in sequence (3) are solved with the LU factorization.

First, assume that $A$ is an $m \times n$ matrix that can be row reduced to echelon form *without row swaps*. Then $A$ can be written in the form $A = LU$ where $L$ is an $m \times m$ lower triangular matrix with 1s on the diagonal and $U$ is an $m \times n$ echelon form of $A$:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} \square & * & * & * & * \\ 0 & \square & * & * & * \\ 0 & 0 & 0 & \square & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is called an *LU factorization* of $A$. The matrix $L$ is invertible and is called a unit lower triangular matrix.

Before giving an algorithm for finding $L$ and $U$, we look at why this decomposition is useful. When $A = LU$, the equation $A\mathbf{x} = \mathbf{b}$ can be written as $L(U\mathbf{x}) = \mathbf{b}$. Then writing $\mathbf{y} := U\mathbf{x}$, we can find $\mathbf{x}$ by solving the following pair of equations:

$$\begin{aligned} L\mathbf{y} &= \mathbf{b} \\ U\mathbf{x} &= \mathbf{y}. \end{aligned}$$

First solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$ and then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$. Each equation is easy to solve because $L$ and $U$ are triangular.

**Example 4.** *It can be checked that*

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} = LU.$$

*Use this LU factorization of A to solve* $A\mathbf{x} = \mathbf{b}$, *where* $\mathbf{b} = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix}$.

The computational efficiency of the LU factorization depends on knowing $L$ and $U$. The algorithm for finding the LU factorization shows that the row reduction of $A$ to an echelon form $U$ amounts to an LU factorization because it produces $L$ with essentially no extra work.

Suppose $A$ can be reduced to an echelon form $U$ using only row replacements that add a multiple of one row to another row *below* it. In this case, there exist unit lower triangular elementary matrices $E_1, \dots E_p$ such that

$$E_p \cdots E_1 A = U. \tag{4}$$

Then

$$A = (E_p \cdots E_1)^{-1} U = LU \tag{5}$$

where $L = (E_p \cdots E_1)^{-1}$. It can be shown that products and inverses of unit lower triangular matrices are also unit lower triangular. Thus $L$ is unit lower triangular.

Note that the row operations that reduce $A$ to $U$ in (4) also reduce the $L$ to $I$ in (5), because

$$E_p \cdots E_1 L = (E_p \cdots E_1)(E_p \cdots E_1)^{-1} = I.$$

This observation is the key to constructing $L$.

Algorithm for an LU factorization

1. Reduce $A$ to an echelon form $U$ by a sequence of row replacement operations, if possible.
2. Place entries in $L$ such that the same sequence of row operations reduces $L$ to $I$.

Step 1 is not always possible, but when it is, by the argument above, this implies that an LU factorization exists. In the next example, we will demonstrate how to implement Step 2.

**Example 5** (2.5.7)**.** *Find an LU factorization of* $A = \begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix}$.

**Example 6.** *Find an LU factorization of* $A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$.