What is on today

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1 Introduction to Determinants

Lay–Lay–McDonald §3.1 pp. 166 – 169

Recall that we saw that a $2 \times 2$ matrix is invertible if and only if its determinant is nonzero. To extend this useful fact to larger matrices, we need a definition for the determinant of an $n \times n$ matrix. Let $A = [a_{ij}]$ be an $n \times n$ matrix.

When $n = 1$, we define $\det A = a_{11}$.

Recall that when $n = 2$, that is, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we have

$$\det A = a_{11}a_{22} - a_{12}a_{21}.$$ 

When $n = 3$, the determinant $\det A$ is defined recursively using determinants of $2 \times 2$ submatrices. That is, suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then

$$\det A = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$ 

For brevity, we write this as

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13},$$ 

where $A_{11}, A_{12},$ and $A_{13}$ are obtained from $A$ by deleting the first row and one of the three columns. For any square matrix $A$, let $A_{ij}$ denote the submatrix formed by deleting the $i$th row and $j$th column of $A$. Now we can give a recursive definition of determinants. When $n = 4$, $\det A$ uses determinants of $3 \times 3$ submatrices, and in general, the determinant of an $n \times n$ matrix is computed using determinants of $(n - 1) \times (n - 1)$ submatrices.
Definition 1. For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is the following:

$$\det A = \sum_{j=1}^{n} (-1)^{1+j}a_{1j} \det A_{1j}.$$ 

Example 2. Compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

To state the next theorem, it is convenient to write the definition of $\det A$ in a slightly different form. Given $A = [a_{ij}]$, the $(i,j)$-cofactor of $A$ is the number $C_{ij}$ given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$ 

Then the formula we just wrote is

$$\det A = a_{i1}C_{11} + a_{i2}C_{12} + \cdots + a_{in}C_{1n}.$$ 

This formula is called a cofactor expansion across the first row of $A$.

Theorem 3. The determinant of an $n \times n$ matrix $A$ can be computed by a cofactor expansion across any row or column. The expansion across the $i$th row is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$ 

The expansion down the $j$th column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$ 

The theorem tells us that we have some flexibility in computing the determinant: by picking a favorable row or column (e.g., one with many zeros), we can cut down on the number of computations we have to do.
Example 4. Compute $\det A$, where $A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$.

The previous example motivates the following useful result:

**Theorem 5.** If $A$ is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of $A$.

## 2 Properties of Determinants

Lay–Lay–McDonald §3.2 pp. 171 – 177

The properties of determinants are governed by row operations. Here are some useful results:

**Theorem 6.** Let $A$ be a square matrix.

1. If a multiple of one row of $A$ is added to another row to produce a matrix $B$, then $\det B = \det A$.

2. If two rows of $A$ are swapped to produce $B$, then $\det B = -\det A$.

3. If one row of $A$ is multiplied by $k$ to produce $B$, then $\det B = k \cdot \det A$.

Suppose a square matrix $A$ has been reduced to an echelon form $U$ by row replacements and row swaps. (This is always possible by the row reduction algorithm.) If there are $r$ swaps, the previous theorem tells us that

$$\det A = (-1)^r \det U.$$ 

Moreover, since $U$ is in echelon form, it is triangular, and so $\det U$ is the product of the diagonal entries $u_{ii}$. If $A$ is invertible, the entries $u_{ii}$ are all pivots (because $A \sim I_n$ and the $u_{ii}$ have not been scaled to 1s). Otherwise, at least $u_{nn}$ will be zero, and the product of diagonal entries will be 0. This gives us

$$\det A = \begin{cases} (-1)^r \cdot \text{(product of pivots in } U) & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible.} \end{cases}$$

The formula above proves the following theorem:
Theorem 7. A square matrix $A$ is invertible if and only if $\det A \neq 0$.

Example 8. Compute $\det A$, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$.

Here are some further useful properties of determinants:

Theorem 9. If $A$ is an $n \times n$ matrix, then $\det A^T = \det A$.

Theorem 10. If $A$ and $B$ are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

3 Cramer’s Rule, Volume, and Linear Transformations

Today we give some formulas for using the determinant in various calculations.

Cramer’s Rule can be used to study how the solution of $Ax = b$ changes as the entries of $b$ change. To give the rule, we first define some notation:

For any $n \times n$ matrix $A$ and any vector $b \in \mathbb{R}^n$, let $A_i(b)$ be the matrix obtained from $A$ by replacing column $i$ by the vector $b$:

$$A_i(b) = [a_1 \cdots b \cdots a_n],$$

where $b$ takes the place of $a_i$.

Theorem 11 (Cramer’s rule). Let $A$ be an invertible $n \times n$ matrix. For any $b \in \mathbb{R}^n$, the unique solution $x$ of $Ax = b$ has entries given by

$$x_i = \frac{\det A_i(b)}{\det A}, \quad i = 1, 2, \ldots, n.$$
Example 12. Use Cramer’s rule to solve the system

\[\begin{align*}
3x_1 - 2x_2 &= 6 \\
-5x_1 + 4x_2 &= 8.
\end{align*}\]

Cramer’s rule also gives us a formula for the inverse of an \(n \times n\) matrix \(A^{-1}\). The \(j\)th column of \(A^{-1}\) is a vector \(\mathbf{x}\) that satisfies \(A\mathbf{x} = \mathbf{e}_j\), where \(\mathbf{e}_j\) is the \(j\)th column of the identity matrix, and the \(i\)th entry of \(\mathbf{x}\) is the \((i, j)\)th entry of \(A^{-1}\). By Cramer’s rule, we have

\[(i, j)\text{th entry of } A^{-1} = x_i = \frac{\det A_i(\mathbf{e}_j)}{\det A}.\]

Recall that \(A_{ji}\) denotes the submatrix of \(A\) formed by deleting row \(j\) and column \(i\). A cofactor expansion down column \(i\) of \(A_i(\mathbf{e}_j)\) shows that

\[\det A_i(\mathbf{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji},\]

where \(C_{ji}\) is a cofactor of \(A\). Thus we have the following formula:

\[A^{-1} = \frac{1}{\det A} \begin{bmatrix}
C_{11} & C_{21} & \cdots & C_{n1} \\
C_{12} & C_{22} & \cdots & C_{n2} \\
\vdots & \vdots & \ddots & \vdots \\
C_{1n} & C_{2n} & \cdots & C_{nn}
\end{bmatrix}.\]  \hspace{1cm} (1)

The matrix of cofactors on the right side of (1) is the adjugate of \(A\), denoted by \(\text{adj } A\).

Theorem 13. Let \(A\) be an invertible \(n \times n\) matrix. Then

\[A^{-1} = \frac{1}{\det A} \text{adj } A.\]

Example 14. Find the inverse of the matrix

\[A = \begin{bmatrix}
2 & 1 & 3 \\
1 & -1 & 1 \\
1 & 4 & -2
\end{bmatrix}.\]