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What is on today

1 Introduction to Determinants

Lay–Lay–McDonald §3.1 pp. 166 – 169

Recall that we saw that a 2×2 matrix is invertible if and only if its determinant is nonzero. To extend this useful fact to larger matrices, we need a definition for the determinant of an $n \times n$ matrix. Let $A = [a_{ij}]$ be an $n \times n$ matrix.

When $n = 1$, we define $\det A = a_{11}$.

Recall that when $n = 2$, that is, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we have

$$\det A = a_{11}a_{22} - a_{12}a_{21}.$$

When $n = 3$, the determinant $\det A$ is defined recursively using determinants of 2×2 submatrices. That is, suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then

$$\det A = {}^{+1}a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

note: signs alternate

For brevity, we write this as

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13},$$

where A_{11}, A_{12}, and A_{13} are obtained from A by deleting the first row and one of the three columns. For any square matrix A, let A_{ij} denote the submatrix formed by deleting the ith row and jth column of A. Now we can give a recursive definition of determinants. When $n = 4$, $\det A$ uses determinants of 3×3 submatrices, and in general, the determinant of an $n \times n$ matrix is computed using determinants of $(n-1) \times (n-1)$ submatrices.

Definition 1. *For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is the following:*

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}. \qquad (*)$$

Example 2. *Compute the determinant of* $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

$$\det A = (-1)^{1+1} \cdot (1) \cdot \det \begin{pmatrix} 4 & -1 \\ -2 & 0 \end{pmatrix} + (-1)^{1+2} \cdot 5 \cdot \det \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} + (-1)^{1+3} \cdot 0 \cdot \det \begin{pmatrix} 2 & 4 \\ 0 & -2 \end{pmatrix}$$

$$= \det \begin{pmatrix} 4 & -1 \\ -2 & 0 \end{pmatrix} + 0 + 0$$

$$= 0 - 2$$

$$= -2$$

now, along 3rd column:

$$\det A = \cancel{(-1)^{1+3} \cdot 0 \cdot \det \begin{pmatrix} 2 & 4 \\ 0 & -2 \end{pmatrix}} + (-1)^{2+3} (-1) \cdot \det \begin{pmatrix} 1 & 5 \\ 0 & -2 \end{pmatrix} + \cancel{(-1)^{3+3} \cdot 0 \cdot \det (\;)} = (-1)(-1)(-2) = -2$$

using 3rd row:

$$\det A = \cancel{(-1)^{3+1} \cdot 0 \cdot \det (\;)} + (-1)^{3+2} \cdot (-2) \det \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} + \cancel{(-1)^{3+3} \cdot 0 \cdot \det (-)} = -1 \cdot (-2)(-1) = -2$$

To state the next theorem, it is convenient to write the definition of $\det A$ in a slightly different form. Given $A = [a_{ij}]$, the *(i,j)-cofactor* of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

Then the formula we just wrote is

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}. \qquad (*)$$

This formula is called a *cofactor expansion across the first row of A.*

Theorem 3. *The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across* any *row or column. The expansion across the ith row is*

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

The expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

The theorem tells us that we have some flexibility in computing the determinant: by picking a favorable row or column (e.g., one with many zeros), we can cut down on the number of computations we have to do.

let's redo Ex 2 above (in blue)

Example 4. *Compute* $\det A$*, where* $A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$.

$$\det A = (-1)^{1+1}(3) \cdot \begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} = 3(2)(-1)^{1+1} \cdot \underbrace{\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}}_{\substack{\text{det already computed} \\ \text{in Ex. 2} \\ = -2}}$$

$$= 3 \cdot 2 \cdot (-2) = -12$$

The previous example motivates the following useful result:

Theorem 5. *If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.*

2 Properties of Determinants

Lay–Lay–McDonald §3.2 pp. 171 – 177

The properties of determinants are governed by row operations. Here are some useful results:

Theorem 6. *Let A be a square matrix.*

1. *If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.*
2. *If two rows of A are swapped to produce B, then $\det B = -\det A$.*
3. *If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.*

Suppose a square matrix A has been reduced to an echelon form U by row replacements and row swaps. (This is always possible by the row reduction algorithm.) If there are r swaps, the previous theorem tells us that

$$\det A = (-1)^r \det U.$$

Moreover, since U is in echelon form, it is triangular, and so $\det U$ is the product of the diagonal entries u_{ii}. If A is invertible, the entries u_{ii} are all pivots (because $A \sim I_n$ and the u_{ii} have not been scaled to 1s). Otherwise, at least u_{nn} will be zero, and the product of diagonal entries will be 0. This gives us

$$\det A = \begin{cases} (-1)^r \cdot (\text{product of pivots in } U) & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible.} \end{cases}$$

The formula above proves the following theorem:

Theorem 7. *A square matrix A is invertible if and only if* $\det A \neq 0$.

Example 8. *Compute* $\det A$*, where* $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$.

$2 \cdot \#1 + \#3$ gives ↓ (this gives B with $\det B = \det A$)

$B = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{bmatrix}$ this means B is not invertible.

since row 2 = row 3 $\Rightarrow \det B = 0$

$\Rightarrow \det A = 0$.

Here are some further useful properties of determinants:

Theorem 9. *If A is an $n \times n$ matrix, then* $\det A^T = \det A$.

Theorem 10. *If A and B are $n \times n$ matrices, then* $\det AB = (\det A)(\det B)$.

3 Cramer's Rule, Volume, and Linear Transformations

Lay–Lay–McDonald §3.3 pp. 179 – 186

Today we give some formulas for using the determinant in various calculations.

Cramer's Rule can be used to study how the solution of $A\mathbf{x} = \mathbf{b}$ changes as the entries of $\mathbf{b}$ change. To give the rule, we first define some notation:

For any $n \times n$ matrix A and any vector $\mathbf{b} \in \mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i by the vector $\mathbf{b}$:

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \cdots \underset{\substack{\uparrow \\ \text{ith col.}}}{\mathbf{b}} \cdots \mathbf{a}_n],$$

where $\mathbf{b}$ takes the place of $\mathbf{a}_i$.

Theorem 11 (Cramer's rule)**.** *Let A be an invertible $n \times n$ matrix. For any* $\mathbf{b} \in \mathbb{R}^n$*, the unique solution* $\mathbf{x}$ *of* $A\mathbf{x} = \mathbf{b}$ *has entries given by*

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, 2, \dots n.$$

Example 12. *Use Cramer's rule to solve the system* $A = \begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix}$ $b = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8. \end{aligned}$$

$$x_1 = \frac{\det\begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}}{\det\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}} = \frac{24+16}{12-10} = \frac{40}{2} = 20$$

$$x_2 = \frac{\det\begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}}{\det\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}} = \frac{24+30}{12-10} = \frac{54}{2} = 27$$

$x_1 = 20$

$x_2 = 27$

Cramer's rule also gives us a formula for the inverse of an $n \times n$ matrix A^{-1}. The jth column of A^{-1} is a vector $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{e}_j$, where $\mathbf{e}_j$ is the jth column of the identity matrix, and the ith entry of $\mathbf{x}$ is the (i,j)th entry of A^{-1}. By Cramer's rule, we have

$$(i,j)\text{th entry of } A^{-1} = x_i = \frac{\det A_i(\mathbf{e}_j)}{\det A}.$$

Recall that A_{ji} denotes the submatrix of A formed by deleting row j and column i. A cofactor expansion down column i of $A_i(\mathbf{e}_j)$ shows that

$$\det A_i(\mathbf{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji},$$

where C_{ji} is a cofactor of A. Thus we have the following formula:

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}. \tag{1}$$

The matrix of cofactors on the right side of (1) is the *adjugate* of A, denoted by adj A.

Theorem 13. *Let A be an invertible $n \times n$ matrix. Then*

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A.$$

$$A^{-1} = \frac{1}{14} \cdot \begin{pmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{pmatrix} = \boxed{\begin{pmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{pmatrix}}$$

Example 14. *Find the inverse of the matrix* $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$.

$C_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix}$ $C_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$ $C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix}$

$C_{21} = -1 \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix}$ $C_{22} = (+1) \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}$ $C_{23} = (-1) \cdot \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix}$

$C_{31} = +1 \cdot \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}$ $C_{32} = (-1) \cdot \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}$ $C_{33} = (+1) \cdot \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$

$$\operatorname{adj} A = \begin{pmatrix} -2 & 3 & 5 \\ 14 & -7 & -7 \\ 4 & 1 & -3 \end{pmatrix}^T = \begin{pmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{pmatrix}.$$

$A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A$

$A^{-1} \cdot A = \left(\frac{1}{\det A}\right) \cdot (\operatorname{adj} A) \cdot A$

$(\det A) \cdot I = (\operatorname{adj} A) \cdot A$

Take $(\operatorname{adj} A) \cdot (A)$

$= \begin{pmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{pmatrix}$

$= \begin{pmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{pmatrix} = 14 \cdot I$

$\det A = 14$