What is on today

1. Cramer’s Rule, Volume, and Linear Transformations
2. Vector Spaces and Subspaces

1. Cramer’s Rule, Volume, and Linear Transformations

Lay–Lay–McDonald §3.3 pp. 179 – 186

Today we give a geometric interpretation of the determinant.

**Theorem 1.** If $A$ is a $2 \times 2$ matrix, the area of the parallelogram determined by the columns of $A$ is $|\det A|$. If $A$ is a $3 \times 3$ matrix, the volume of the parallelepiped determined by the columns of $A$ is $|\det A|$.

**Example 2.** Calculate the area of the parallelogram determined by the points $(-2, -2), (0, 3), (4, -1)$, and $(6, 4)$.

Determinants can be used to describe an important geometric property of linear transformations in the plane and in $\mathbb{R}^3$. If $T$ is a linear transformation and $S$ is a set in the domain of $T$, let $T(S)$ denote the set of images of points in $S$. We are interested in how the area (or volume) of $T(S)$ compares with the area (or volume) of the original set $S$.

**Theorem 3.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a $2 \times 2$ matrix $A$. If $S$ is a parallelogram in $\mathbb{R}^2$, then

$$\text{area of } T(S) = |\det A| \cdot \text{area of } S.$$  

If $T$ is determined by a $3 \times 3$ matrix $A$, and if $S$ is a parallelepiped in $\mathbb{R}^3$, then

$$\text{volume of } T(S) = |\det A| \cdot \text{volume of } S.$$  

It turns out that the conclusions of the above theorem hold whenever $S$ is a region in $\mathbb{R}^2$ with finite area or a region in $\mathbb{R}^3$ with finite volume.
Example 4. Let $a$ and $b$ be positive numbers. Find the area of the region $E$ bounded by the ellipse whose equation is 
\[
\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.
\]

2 Vector Spaces and Subspaces

Lay–Lay–McDonald §4 pp. 192 – 197

The work we’ve been doing with vectors in $\mathbb{R}^n$ can be understood in a more general framework once we have the notion of a vector space, which will be our object of study today.

Definition 5. A (real) vector space is a nonempty set $V$ of objects, called vectors, on which are defined two operations, addition and multiplication by scalars (real numbers), subject to the ten axioms below. The axioms must hold for all vectors $u, v, w \in V$ and for all scalars $c, d$.

1. $u + v \in V$.
2. $u + v = v + u$.
3. $(u + v) + w = u + (v + w)$.
4. There is a zero vector $0 \in V$ such that $u + 0 = u$.
5. For each $u \in V$, there is a vector $-u \in V$ such that $u + (-u) = 0$.
6. $cu \in V$.
7. $c(u + v) = cu + cv$.
8. $(c + d)u = cu + du$.
9. $c(du) = (cd)u$.
10. $1u = u$.

Note that the zero vector $0$ is unique, and for each $u \in V$, its negative $-u$ is unique.
Example 6. The spaces $\mathbb{R}^n$ for $n \geq 1$ are vector spaces.

Example 7. For $n \geq 0$, the set $P_n$ of polynomials of degree at most $n$ consists of all polynomials of the form

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n,$$

where the coefficients $a_0, a_1, \ldots, a_n$ are real numbers. If $p(t) = a_0 \neq 0$, the degree of $p$ is zero. If all of the coefficients are zero, $p$ is called the zero polynomial. Show that $P_n$ is a vector space.

Example 8. Let $V$ be the set of all real-valued functions defined on a set $D$ (where $D$ is $\mathbb{R}$ or some interval on the real line). Show that $V$ is a vector space.

In many problems, a vector space consists of a subset of vectors from some larger vector space. In this case, only three of the ten vector space axioms need to be checked; the rest are automatically satisfied.

Definition 9. A subspace of a vector space $V$ is a subset $H$ of $V$ that has three properties:

1. The zero vector of $V$ is in $H$.
2. $H$ is closed under vector addition: $u, v \in H \Rightarrow u + v \in H$.
3. $H$ is closed under multiplication by scalars: if $c$ is a scalar and $u \in H$, then $cu \in H$.

Example 10. Is the set consisting of the zero vector in a vector space $V$ a subspace of $V$?
Example 11. Let $P$ be the set of all polynomials with real coefficients, with the usual operations in $P$. Then $P$ is a subspace of the space of all real-valued functions on $\mathbb{R}$. Also, for each $n \geq 0$, $P_n$ is a subspace of $P$.

Example 12. The vector space $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$, since $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$. However, the set $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} \right\}$ where $s, t \in \mathbb{R}$ is a subset of $\mathbb{R}^3$. Show that $H$ is a subspace of $\mathbb{R}^3$.

Example 13. Consider a plane in $\mathbb{R}^3$ not through the origin. Is it a subspace of $\mathbb{R}^3$?

Example 14. Let $V$ be a vector space, and let $v_1, v_2 \in V$. Let $H = \text{Span}\{v_1, v_2\}$ be the set of all linear combinations of $v_1, v_2$. Show that $H$ is a subspace of $V$.

The argument in the previous example can be generalized to prove the following:

Theorem 15. If $v_1, \ldots, v_p$ are in a vector space $V$, then $\text{Span}\{v_1, \ldots, v_p\}$ is a subspace of $V$.

We call $\text{Span}\{v_1, \ldots, v_p\}$ the subspace spanned (or generated) by $\{v_1, \ldots, v_p\}$. Given any subspace $H$ of $V$, a spanning (or generating) set for $H$ is a set $\{v_1, \ldots, v_p\}$ in $H$ such that $H = \text{Span}\{v_1, \ldots, v_p\}$.  

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Example 16. Let $H$ be the set of all vectors of the form $(a - 3b, b - a, a, b)$, where $a, b$ are arbitrary real numbers. Show that $H$ is a subspace of $\mathbb{R}^4$.

Example 17. For what values of $h$ will $y$ be in the subspace of $\mathbb{R}^3$ spanned by $v_1, v_2, v_3$ if

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, y = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}?$$

Example 18. Show that the set $H$ of all points of $\mathbb{R}^2$ of the form $(3a, 2 + 5a)$ is not a vector space.