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What is on today

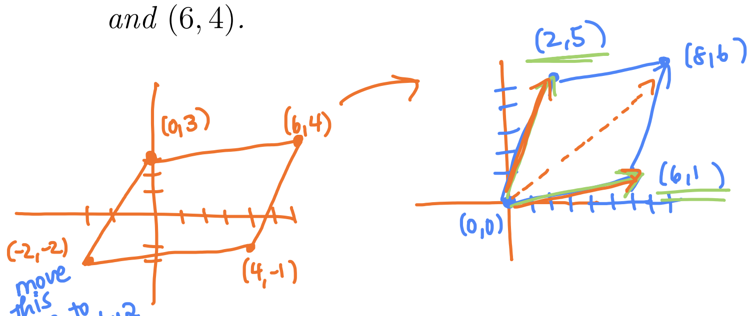
1 Cramer's Rule, Volume, and Linear Transformations

Lay–Lay–McDonald §3.3 pp. 179 – 186

Today we give a geometric interpretation of the determinant.

Theorem 1. *If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.*

Example 2. *Calculate the area of the parallelogram determined by the points* $(-2,-2), (0,3), (4,-1)$, *and* $(6,4)$.



move this up 2, to the right by 2

$(-2,-2)\to(0,0)$

these give the sides of parallelogram

⇝ take vectors $\begin{pmatrix}2\\5\end{pmatrix}, \begin{pmatrix}6\\1\end{pmatrix}$, put them in matrix:

$$\text{abs}\left(\begin{vmatrix}2 & 6\\ 5 & 1\end{vmatrix}\right) = \text{abs}(2-30) = |-28| = 28$$

Determinants can be used to describe an important geometric property of linear transformations in the plane and in $\mathbb{R}^3$. If T is a linear transformation and S is a set in the domain of T, let $T(S)$ denote the set of images of points in S. We are interested in how the area (or volume) of $T(S)$ compares with the area (or volume) of the original set S.

Theorem 3. *Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then*

$$\text{area of } T(S) = |\det A|\cdot \text{area of } S.$$

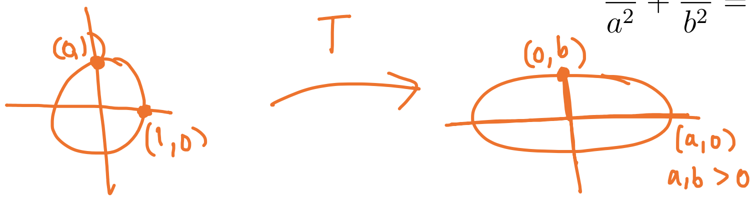
If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\text{volume of } T(S) = |\det A|\cdot \text{volume of } S.$$

It turns out that the conclusions of the above theorem hold whenever S is a region in $\mathbb{R}^2$ with finite area or a region in $\mathbb{R}^3$ with finite volume.

Example 4. *Let a and b be positive numbers. Find the area of the region E bounded by the ellipse whose equation is*

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.$$



standard matrix:

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$\text{area}(E) = \left|\det\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right| \cdot \text{area(circle)} = ab \cdot \pi \cdot 1^2 = \pi \cdot ab$$

2 Vector Spaces and Subspaces

Lay–Lay–McDonald §4.1 pp. 192 – 197

The work we've been doing with vectors in $\mathbb{R}^n$ can be understood in a more general framework once we have the notion of a *vector space*, which will be our object of study today.

Definition 5. *A (real)* vector space *is a nonempty set V of objects, called vectors, on which are defined two operations, addition and multiplication by scalars (real numbers), subject to the ten axioms below. The axioms must hold for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and for all scalars c, d.*

1. $\mathbf{u} + \mathbf{v} \in V$.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. *There is a zero vector* $\mathbf{0} \in V$ *such that* $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. *For each* $\mathbf{u} \in V$*, there is a vector* $-\mathbf{u} \in V$ *such that* $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u} \in V$.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.

Note that the zero vector $\mathbf{0}$ is unique, and for each $\mathbf{u} \in V$, its negative $-\mathbf{u}$ is unique.

Example 6. *The spaces* $\mathbb{R}^n$ *for* $n \geq 1$ *are vector spaces.*

Example 7. *For* $n \geq 0$*, the set* P_n *of polynomials of degree at most* n *consists of all polynomials of the form*

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n,$$

where the coefficients $a_0, a_1, \ldots, a_n$ *are real numbers. If* $p(t) = a_0 \neq 0$*, the degree of* p *is zero. If all of the coefficients are zero,* p *is called the zero polynomial. Show that* P_n *is a vector space.*

what do two vectors look like?

$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$

$q(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n$

want: if $p, q \in P_n$, then $p+q \in P_n$.

$p(t) + q(t) = (a_0 + a_1 t + \cdots + a_n t^n) + (b_0 + b_1 t + \cdots + b_n t^n)$

$= (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n$ ✓ $\in P_n$

$c \cdot p(t) = c(a_0 + a_1 t + \cdots + a_n t^n)$

$= ca_0 + ca_1 t + \cdots + ca_n t^n \in P_n$ ✓

other properties follow from properties of real numbers & polynomials w/ real coeffs. $\vec{0}$ is 0 polynomial

$-p = -a_0 - a_1 t - \cdots - a_n t^n$.

$\Rightarrow P_n$ is a vector space.

Example 8. *Let* V *be the set of all real-valued functions defined on a set* D *(where* D *is* $\mathbb{R}$ *or some interval on the real line). Show that* V *is a vector space.*

$f, g \in V \Rightarrow f + g \in V$

$f(t), t \in D$, $(f+g)(t) = \overbrace{\underbrace{f(t)}_{\text{on } D} + \underbrace{g(t)}_{\text{on } D}}^{\text{on } D} \in V$ ✓

this is still a real-valued function on D

scalar multiple: $cf(t) = c \cdot f(t)$

real-valued, defined on D, so is cf.

zero vector: $f(t) = 0$ on all of D

negative of f is $-f$; other axioms follow from properties of real #s.

In many problems, a vector space consists of a subset of vectors from some larger vector space. In this case, only three of the ten vector space axioms need to be checked; the rest are automatically satisfied.

Definition 9. *A* subspace *of a vector space* V *is a subset* H *of* V *that has three properties:*

1. *The zero vector of* V *is in* H*.*
2. H *is closed under vector addition:* $\mathbf{u}, \mathbf{v} \in H \Rightarrow \mathbf{u} + \mathbf{v} \in H$*.*
3. H *is closed under multiplication by scalars: if* c *is a scalar and* $\mathbf{u} \in H$*, then* $c\mathbf{u} \in H$*.*

Example 10. *Is the set consisting of the zero vector in a vector space* V *a subspace of* V*?*

$Z = \{\vec{0}\}$

is zero vector in Z? ✓

closed under vector addition ✓

closed under scalar mult. ✓

$\Rightarrow$ Z is a subspace of V.

Example 11. *Let P be the set of all polynomials with real coefficients, with the usual operations in P. Then P is a subspace of the space of all real-valued functions on $\mathbb{R}$. Also, for each $n \geq 0$, P_n is a subspace of P.*

try this at home!

Example 12. *The vector space $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$, since $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$. However, the set $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} \right\}$ where $s, t \in \mathbb{R}$ is a subset of $\mathbb{R}^3$. Show that H is a subspace of $\mathbb{R}^3$.*

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in H$ ✓

If $v_1 = \begin{pmatrix} s_1 \\ t_1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} s_2 \\ t_2 \\ 0 \end{pmatrix}$ $\Rightarrow v_1 + v_2 = \begin{pmatrix} s_1 + s_2 \\ t_1 + t_2 \\ 0 \end{pmatrix} \in H$ ✓

$c \cdot v_1 = \begin{pmatrix} cs_1 \\ ct_1 \\ 0 \end{pmatrix} \in H$ ✓

So H is a subspace of $\mathbb{R}^3$.

Example 13. *Consider a plane in $\mathbb{R}^3$ not through the origin. Is it a subspace of $\mathbb{R}^3$?*

The zero vector in $\mathbb{R}^3$ is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. This plane doesn't contain the zero vector.

So no! Not a subspace of $\mathbb{R}^3$.

Example 14. *Let V be a vector space, and let $\mathbf{v}_1, \mathbf{v}_2 \in V$. Let $H =$Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ be the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2$. Show that H is a subspace of V.*

$0 \in H$? $\quad 0 = 0 \cdot v_1 + 0 \cdot v_2 \in H$ ✓

$v_1 + v_2 \in H$ ✓

$= 1 \cdot v_1 + 1 v_2$

$cv_1 \in H$ ✓

$\Rightarrow$ So H is a subspace of V.

The argument in the previous example can be generalized to prove the following:

Theorem 15. *If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.*

We call Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ the subspace spanned (or generated) by $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Given any subspace H of V, a spanning (or generating) set for H is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in H such that $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Example 16. *Let H be the set of all vectors of the form $(a-3b, b-a, a, b)$, where a, b are arbitrary real numbers. Show that H is a subspace of $\mathbb{R}^4$.*

$$\begin{pmatrix} a-3b \\ b-a \\ a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b \cdot \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\} \Rightarrow H \text{ is subspace of } \mathbb{R}^4.$$

Example 17. *For what values of h will $\mathbf{y}$ be in the subspace of $\mathbb{R}^3$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ if*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}?$$

when is $y = a_1 v_1 + a_2 v_2 + a_3 v_3$?

$a_1 \cdot 1 + a_2 \cdot 5 + a_3 \cdot -3 = -4$

$a_1 \cdot -1 + a_2 \cdot -4 + a_3 \cdot 1 = 3$

$a_1 \cdot -2 + a_2 \cdot -7 + a_3 \cdot 0 = h$

$$\Rightarrow \begin{pmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & 7 & 0 & h \end{pmatrix} \sim \begin{pmatrix} 1 & 5 & -3 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & -8+h \end{pmatrix} \sim \begin{pmatrix} 1 & 5 & -3 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & -5+h \end{pmatrix}$$

$\Rightarrow h = 5.$

Example 18. *Show that the set H of all points of $\mathbb{R}^2$ of the form $(3a, 2+5a)$ is not a vector space.*

try this at home!