What is on today

1 Cramer’s Rule, Volume, and Linear Transformations
2 Vector Spaces and Subspaces

1 Cramer’s Rule, Volume, and Linear Transformations

Lay–Lay–McDonald §3.3 pp. 179 – 186

Today we give a geometric interpretation of the determinant.

**Theorem 1.** If $A$ is a $2 \times 2$ matrix, the area of the parallelogram determined by the columns of $A$ is $|\det A|$. If $A$ is a $3 \times 3$ matrix, the volume of the parallelepiped determined by the columns of $A$ is $|\det A|$.

**Example 2.** Calculate the area of the parallelogram determined by the points $(-2, -2), (0, 3), (4, -1)$, and $(6, 4)$.

Determinants can be used to describe an important geometric property of linear transformations in the plane and in $\mathbb{R}^3$. If $T$ is a linear transformation and $S$ is a set in the domain of $T$, let $T(S)$ denote the set of images of points in $S$. We are interested in how the area (or volume) of $T(S)$ compares with the area (or volume) of the original set $S$.

**Theorem 3.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a $2 \times 2$ matrix $A$. If $S$ is a parallelogram in $\mathbb{R}^2$, then

$$\text{area of } T(S) = |\det A| \cdot \text{area of } S.$$ 

If $T$ is determined by a $3 \times 3$ matrix $A$, and if $S$ is a parallelepiped in $\mathbb{R}^3$, then

$$\text{volume of } T(S) = |\det A| \cdot \text{volume of } S.$$ 

It turns out that the conclusions of the above theorem hold whenever $S$ is a region in $\mathbb{R}^2$ with finite area or a region in $\mathbb{R}^3$ with finite volume.
Example 4. Let \( a \) and \( b \) be positive numbers. Find the area of the region \( E \) bounded by the ellipse whose equation is
\[
\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.
\]

\[\begin{array}{c}
\text{standard matrix:} \\
\begin{pmatrix}
a & 0 \\
0 & b
\end{pmatrix}
\end{array}\]

\[\text{area } (E) = |\det \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}| \cdot \text{area(circle)} = ab \cdot \pi \cdot 1^2 = \pi ab\]

2 Vector Spaces and Subspaces

Lay–Lay–McDonald §4.1 pp. 192 – 197

The work we’ve been doing with vectors in \( \mathbb{R}^n \) can be understood in a more general framework once we have the notion of a vector space, which will be our object of study today.

Definition 5. A (real) vector space is a nonempty set \( V \) of objects, called vectors, on which are defined two operations, addition and multiplication by scalars (real numbers), subject to the ten axioms below. The axioms must hold for all vectors \( u, v, w \in V \) and for all scalars \( c, d \).

1. \( u + v \in V \).
2. \( u + v = v + u \).
3. \( (u + v) + w = u + (v + w) \).
4. There is a zero vector \( 0 \in V \) such that \( u + 0 = u \).
5. For each \( u \in V \), there is a vector \(-u \in V \) such that \( u + (-u) = 0 \).
6. \( cu \in V \).
7. \( c(u + v) = cu + cv \).
8. \( (c+d)u = cu + du \).
9. \( c(du) = (cd)u \).
10. \( 1u = u \).

Note that the zero vector \( 0 \) is unique, and for each \( u \in V \), its negative \(-u \) is unique.
Example 6. The spaces $\mathbb{R}^n$ for $n \geq 1$ are vector spaces.

Example 7. For $n \geq 0$, the set $P_n$ of polynomials of degree at most $n$ consists of all polynomials of the form

$$p(t) = a_0 + a_1t + \cdots + a_nt^n,$$

where the coefficients $a_0, a_1, \ldots, a_n$ are real numbers. If $p(t) = a_0 \neq 0$, the degree of $p$ is zero. If all of the coefficients are zero, $p$ is called the zero polynomial. Show that $P_n$ is a vector space.

Example 8. Let $V$ be the set of all real-valued functions defined on a set $D$ (where $D$ is $\mathbb{R}$ or some interval on the real line). Show that $V$ is a vector space.

In many problems, a vector space consists of a subset of vectors from some larger vector space. In this case, only three of the ten vector space axioms need to be checked; the rest are automatically satisfied.

Definition 9. A subspace of a vector space $V$ is a subset $H$ of $V$ that has three properties:

1. The zero vector of $V$ is in $H$.
2. $H$ is closed under vector addition: $u, v \in H \Rightarrow u + v \in H$.
3. $H$ is closed under multiplication by scalars: if $c$ is a scalar and $u \in H$, then $cu \in H$.

Example 10. Is the set consisting of the zero vector in a vector space $V$ a subspace of $V$?
Example 11. Let $P$ be the set of all polynomials with real coefficients, with the usual operations in $P$. Then $P$ is a subspace of the space of all real-valued functions on $\mathbb{R}$. Also, for each $n \geq 0$, $P_n$ is a subspace of $P$.

Example 12. The vector space $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$, since $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$. However, the set $H = \left\{ \begin{pmatrix} s \\ t \\ 0 \end{pmatrix} \right\}$ where $s, t \in \mathbb{R}$ is a subset of $\mathbb{R}^3$. Show that $H$ is a subspace of $\mathbb{R}^3$.

Example 13. Consider a plane in $\mathbb{R}^3$ not through the origin. Is it a subspace of $\mathbb{R}^3$?

Example 14. Let $V$ be a vector space, and let $v_1, v_2 \in V$. Let $H = \text{Span}\{v_1, v_2\}$ be the set of all linear combinations of $v_1, v_2$. Show that $H$ is a subspace of $V$.

The argument in the previous example can be generalized to prove the following:

Theorem 15. If $v_1, \ldots, v_p$ are in a vector space $V$, then $\text{Span}\{v_1, \ldots, v_p\}$ is a subspace of $V$.

We call $\text{Span}\{v_1, \ldots, v_p\}$ the subspace spanned (or generated) by $\{v_1, \ldots, v_p\}$. Given any subspace $H$ of $V$, a spanning (or generating) set for $H$ is a set $\{v_1, \ldots, v_p\}$ in $H$ such that $H = \text{Span}\{v_1, \ldots, v_p\}$. 

4
Example 16. Let $H$ be the set of all vectors of the form $(a - 3b, b - a, a, b)$, where $a, b$ are arbitrary real numbers. Show that $H$ is a subspace of $\mathbb{R}^4$.

\[
\begin{pmatrix}
a - 3b \\
b - a \\
a \\
b 
\end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\} \Rightarrow H \text{ is subspace of } \mathbb{R}^4.
\]

Example 17. For what values of $h$ will $y$ be in the subspace of $\mathbb{R}^3$ spanned by $v_1, v_2, v_3$ if

\[
v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \quad y = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}.
\]

When is $y = a_1v_1 + a_2v_2 + a_3v_3$?

\[
\begin{align*}
a_1 \cdot 1 + a_2 \cdot 5 + a_3 \cdot -3 &= -4 \Rightarrow \begin{pmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & 7 & 0 & h \end{pmatrix} \\
a_1 \cdot -1 + a_2 \cdot -4 + a_3 \cdot 1 &= 3 \Rightarrow \begin{pmatrix} 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & -8 + h \end{pmatrix} \sim \begin{pmatrix} 1 & 5 & -3 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & -8 + h \end{pmatrix} \sim \begin{pmatrix} 1 & 5 & -3 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & -8 + h \end{pmatrix} \Rightarrow h = 5.
\end{align*}
\]

Example 18. Show that the set $H$ of all points of $\mathbb{R}^2$ of the form $(3a, 2 + 5a)$ is not a vector space.

Try this at home!