Midterm 1 Review — March 9, 2021

Q. Some linear transformations?

$T: \mathbb{R}^2 \to \mathbb{R}^3$

$(x_1, y_1) \to (x_1, x_1, x_1)$.

Is this linear?

Linear transformation:

$T(v_1+v_2) = T(v_1)+T(v_2)$

$T(cv_1) = cT(v_1)$

$v_1 = (x_1, y_1)$

$v_2 = (x_2, y_2)$

$T(v_1+v_2) = T((x_1+x_2, y_1+y_2))$

$= (x_1+x_2, x_1+x_2, x_1+x_2)$

$T(v_1) = (x_1, x_1, x_1)$

$T(v_2) = (x_2, x_2, x_2)$

$T(v_1)+T(v_2) = (x_1+x_2, x_1+x_2, x_1+x_2)$

$= T(v_1+v_2)$ ✓

$T(cv_1) = T((cx_1, cy_1)) = (cx_1, cx_1, cx_1)$

$cT(v_1) = c\cdot(x_1, x_1, x_1) = (cx_1, cx_1, cx_1)$.

✓

Yes, it's linear.

Is this onto?

$T: \mathbb{R}^2 \to \mathbb{R}^3$

$(x_1, y_1) \mapsto (x_1, x_1, x_1)$

e.g. $(1,1) \mapsto (1,1,1)$

$(1,2) \mapsto (1,1,1)$

$(2,3) \longmapsto (2,2,2)$

...

$? \to (1,2,3)$

no, can't get $(1,2,3) \in \mathbb{R}^3$,

so, not onto.

It's not one-to-one because

$(1,1)$, $(1,2)$ $\mapsto (1,1,1)$.

Q. Is $T: \mathbb{R}^2 \to \mathbb{R}^3$

$\vec{x}=(x_1,x_2) \mapsto (2x_1-3x_2,\ x_1+4,\ 5x_2)$ linear?

$\vec{y}=(y_1,y_2) \mapsto (2y_1-3y_2,\ y_1+4,\ 5y_2)$.

$$T(x+y) = T\big((x_1+y_1,\ x_2+y_2)\big)$$
$$= \big(2(x_1+y_1)-3(x_2+y_2),\ x_1+y_1+4,\ 5(x_2+y_2)\big)$$
$$= (2x_1+2y_1\ -3x_2-3y_2,\ x_1+y_1+4,\ 5x_2+5y_2)$$

$$\neq$$

$$T(x)+T(y)$$
$$(\ \text{—},\ x_1+4+y_1+4,\ \text{—}\ )$$
$$x_1+y_1+8$$

not linear!

Q. Suppose A and B are $n \times n$ matrices, B is invertible, and AB is invertible. Show that A is invertible.

⇝ Using determinants:

AB is Invertible $\Rightarrow$ $\det(AB) \neq 0$

Also: $\underbrace{\det(AB)}_{\neq 0} = (\det A)\underbrace{(\det B)}_{\neq 0}$ and B is invertible $\Rightarrow \det B \neq 0$.

$\Rightarrow \det A \neq 0 \Rightarrow$ A is invertible.

$\rightsquigarrow$ Not using determinants:

let $C = AB$; this is invertible by assumption.

Consider $C \cdot B^{-1} = A \cdot B \cdot B^{-1} \Rightarrow A = CB^{-1}$

(this uses that B^{-1} exists)

Now look at $B \cdot C^{-1}$: take $A \cdot (BC^{-1})$ and $(BC^{-1}) \cdot A$

$$A \cdot (BC^{-1}) = (C \cdot B^{-1})(BC^{-1}) = C \cdot (B^{-1}B)C^{-1} = C \cdot C^{-1} = I$$

$$(BC^{-1}) \cdot A = BC^{-1} \cdot CB^{-1} = I$$

So this shows that BC^{-1} is the inverse of A.

So A is invertible.

Q. from chat.

Could imagine $A = 2 \times 3$ matrix, $B = 3 \times 2$ matrix

$AB = 2 \times 2$ matrix & invertible

but A not invertible

$$\text{Span}\{v_1, v_2\} = \{av_1 + bv_2\}$$

$$\text{Span}\{v_1, \ldots, v_n\} = \left\{ \sum_{i=1}^{n} a_i v_i \right\}$$

LU factorization of $A = \begin{pmatrix} 2 & 5 \\ -3 & 4 \end{pmatrix}$

$$= L \cdot U$$

$$= \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

from echelon form.

Compute echelon form of A:

$$\begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix} \underset{-\frac{3}{2}\cdot\#1 + \#2}{\sim} \begin{pmatrix} \boxed{\begin{matrix} 2 \\ 0 \end{matrix}} & \begin{matrix} 5 \\ \frac{23}{2} \end{matrix} \end{pmatrix}$$ ← this is now U (echelon form)

$\div 2$ to get $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 0 \\ \boxed{*} & 1 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 0 & 23/2 \end{pmatrix}$$

$\boxed{*}$ " $-3/2$

used $\div 2$ to get $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so take 1st col. of A and $\div$ by 2 to get 1st col. of L

$$A = \underbrace{\begin{pmatrix} 1 & 0 \\ -3/2 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & 5 \\ 0 & 23/2 \end{pmatrix}}_{U}.$$

2.3 #12c, 12e, 18, 32

#12: ?s about $n\times n$ matrices

12c: If the eqn $Ax=b$ has at least one sol. for each $b\in\mathbb{R}^n$, then the sol. is unique for each b.

True: the corresponding lin. transformation is onto. this is something we can say in terms of the $n\times n$ matrix: this means the transf. is also one-to-one (by Invertible Matrix Thm.) So the sol. is unique for each b.

12e: If there's a $b\in\mathbb{R}^n$ s.t. $Ax=b$ is inconsistent, then $x\mapsto Ax$ is not one-to-one.

True: not onto $\Rightarrow$ by Invertible Matrix Thm $\Rightarrow$ not one-to-one.

18: If C is 6×6 and $Cx=v$ is consistent $\forall\, v\in\mathbb{R}^6$, is it possible that for some v, $Cx=v$ has more than one sol?

square matrix + onto $\Rightarrow$ Invertible Matrix Thm tells us one-to-one.
so, no, can't have more than one sol.

32 : Suppose A is $n \times n$, and $Ax=0$ has only trivial sol.
W/o using IMT, explain why $Ax=b$ must have sol. for each $b \in \mathbb{R}^n$.

$\Rightarrow$ A has a pivot in each of n cols, A is square
$\Rightarrow$ A has a pivot in each row
$\Rightarrow$ $Ax=b$ has a sol. for each $b \in \mathbb{R}^n$.

§1.5. Thm: Suppose $Ax=b$ is consistent for some b, and let p be a solution. Then the solutions to $Ax=b$ is all vectors of the form $w = p + v_h$, where v_h is any sol. of $Ax=0$.

3.3 #32

$[T(e_1) \ T(e_2) \ T(e_3)]$

a) $[v_1 \ \ v_2 \ \ v_3]$

b) vol. of $S = 1/3 \cdot (\text{area of base})(\text{height})$ ✓

vol. of $S' = |\det(T)| \cdot$ vol. of S.