Professor Jennifer Balakrishnan, *jbala@bu.edu*

# What is on today



# 1 Vector spaces and subspaces

Lay–Lay–McDonald §4.1 pp. 192 – 197

Let's recap a few key definitions we saw in this section:

**Definition 1.** *A (real)* vector space *is a nonempty set $V$ of objects, called vectors, on which are defined two operations, addition and multiplication by scalars (real numbers), subject to the ten axioms below. The axioms must hold for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and for all scalars $c, d$.*

1. $\mathbf{u} + \mathbf{v} \in V$.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. *There is a zero vector $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.*
5. *For each $\mathbf{u} \in V$, there is a vector $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.*
6. $c\mathbf{u} \in V$.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.

Recall that in many problems, a vector space consists of a subset of vectors from some larger vector space. In this case, only three of the ten vector space axioms need to be checked; the rest are automatically satisfied.

**Definition 2.** *A* subspace *of a vector space $V$ is a subset $H$ of $V$ that has three properties:*

1. *The zero vector of $V$ is in $H$.*
2. *$H$ is closed under vector addition: $\mathbf{u}, \mathbf{v} \in H \Rightarrow \mathbf{u} + \mathbf{v} \in H$.*
3. *$H$ is closed under multiplication by scalars: if $c$ is a scalar and $\mathbf{u} \in H$, then $c\mathbf{u} \in H$.*

# 2 Null spaces, column spaces, and linear transformations

Lay–Lay–McDonald §4.2 pp. 200 – 207

In applications of linear algebra, subspaces of $\mathbb{R}^n$ usually arise in one of two ways:

1. as the set of all solutions to a system of homogeneous linear equations or
2. as the set of all linear combinations of certain vectors.

Today we work with these two types of subspaces, introduce some new terminology, and recast some material from earlier chapters in this new framework.

Earlier we looked at systems of homogeneous equations like the following:

$$\begin{aligned} x_1 - 3x_2 - 2x_3 &= 0 \\ -5x_1 + 9x_2 + x_3 &= 0. \end{aligned}$$

We can write this in matrix form as $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}.$$

We said that the set of all vectors $\mathbf{x}$ that satisfied the system gave us the *solution set* of the system. In terms of the matrix $A$, we will call the set of $\mathbf{x}$ that satisfy $A\mathbf{x} = \mathbf{0}$ the *null space* of the matrix $A$.

**Definition 3.** *The null space of an $m \times n$ matrix $A$, written as $\operatorname{Nul} A$, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. That is,*

$$\operatorname{Nul} A = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}.$$

**Example 4.** *Let* $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$*, and let* $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$*. Determine if* $\mathbf{u} \in \operatorname{Nul} A$.

It turns out that the null space of a matrix is a vector space!

**Theorem 5.** *The null space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of $m$ homogeneous linear equations in $n$ unknowns is a subspace of $\mathbb{R}^n$.*

*Proof.* First we check that $\operatorname{Nul} A$ is a subset of $\mathbb{R}^n$: $A$ has $n$ columns, so this is clear. Next we show that $\operatorname{Nul} A$ satisfies the three properties of a subspace. The zero vector $\mathbf{0}$ is clearly in $\operatorname{Nul} A$. Next, let $\mathbf{u}, \mathbf{v}$ be two vectors in $\operatorname{Nul} A$. Then $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$, which gives us that

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0},$$

so $\mathbf{u} + \mathbf{v} \in \operatorname{Nul} A$, i.e., $\operatorname{Nul} A$ is closed under vector addition. Finally, if $c$ is any scalar and $\mathbf{u} \in \operatorname{Nul} A$, then

$$A(c\mathbf{u}) = cA\mathbf{u} = c\mathbf{0} = \mathbf{0},$$

and we see that $c\mathbf{u} \in \operatorname{Nul} A$ as well, i.e., $\operatorname{Nul} A$ is closed under scalar multiplication. So $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^n$. $\square$

**Example 6.** *Let $H$ be the set of all vectors in $\mathbb{R}^4$ whose coordinates $a, b, c, d$ satisfy the equations $a - 2b + 5c = d$ and $c - a = b$. Show that $H$ is a subspace of $\mathbb{R}^4$.*

How do we compute $\operatorname{Nul} A$? To compute a spanning set for $\operatorname{Nul} A$, we have to compute a reduced echelon form of $[A \quad \mathbf{0}]$, as we see in the next example.

**Example 7.** *Find a spanning set for the null space of the matrix* $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$.

We make two observations about the above example:

- The spanning set produced by the method above is automatically linearly independent, because the free variables are the weights on the spanning vectors.
- When $\operatorname{Nul} A$ contains nonzero vectors, the number of vectors in the spanning set for $\operatorname{Nul} A$ equals the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.

Another important subspace associated with a matrix is its column space.

**Definition 8.** *The column space of an $m \times n$ matrix $A$, written as $\operatorname{Col} A$, is the set of all linear combinations of the columns of $A$. If $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$, then $\operatorname{Col} A = \operatorname{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$.*

Last week, we saw that $\operatorname{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is a subspace of a vector space; since each column of $A$ is in $\mathbb{R}^m$, we have the following:

**Theorem 9.** *The column space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^m$.*

Note that a typical vector in $\operatorname{Col} A$ can be written as $A\mathbf{x}$ for some $\mathbf{x}$ since $A\mathbf{x}$ gives a linear combination of the columns of $A$. That is, we have

$$\operatorname{Col} A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}.$$

This shows that $\operatorname{Col} A$ is the range of the linear transformation defined by the matrix $A$.

**Example 10.** *Find a matrix $A$ such that $W = \operatorname{Col} A$, where $W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} : a, b \in \mathbb{R} \right\}$.*

**Example 11.** *Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$.*

1. *If the column space of $A$ is a subspace of $\mathbb{R}^k$, what is $k$?*

2. *If the null space of $A$ is a subspace of $\mathbb{R}^k$, what is $k$?*

**Example 12.** *Let* $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$. *Find a nonzero vector in* $\operatorname{Col} A$ *and a nonzero vector in* $\operatorname{Nul} A$.

**Example 13.** *Let* $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, *let* $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$, *and let* $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$.

1. *Is* $\mathbf{u} \in \operatorname{Nul} A$*? Is* $\mathbf{u} \in \operatorname{Col} A$*?*

2. *Is* $\mathbf{v} \in \operatorname{Col} A$*? Is* $\mathbf{v} \in \operatorname{Nul} A$*?*

Now we rephrase things in terms of linear transformations.

Let $A$ be an $m \times n$ matrix.

- $\operatorname{Nul} A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
- $\operatorname{Col} A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

**Definition 14.** *A linear transformation $T$ from a vector space $V$ into a vector space $W$ is a rule that assigns to each vector $\mathbf{x} \in V$ a unique vector $T(\mathbf{x}) \in W$ such that*

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ *for all* $\mathbf{u}, \mathbf{v} \in V$ *and*
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ *for all* $\mathbf{u} \in V$ *and scalars* $c$.

The kernel (or null space) of such a $T$ is the set of all $\mathbf{u} \in V$ such that $T(\mathbf{u}) = \mathbf{0}$. The range of $T$ is the set of all vectors in $W$ of the form $T(\mathbf{x})$ for some $\mathbf{x} \in V$. If $T$ arises as a matrix transformation, e.g., $T(\mathbf{x}) = A\mathbf{x}$ for some matrix $A$, then the kernel and the range of $T$ are just the null space and the column space of $A$ as defined earlier.

**Example 15.** *Let $V$ be the vector space of all real-valued functions $f$ defined on an interval $[a, b]$ with the property that they are differentiable and their derivatives are continuous functions on $[a, b]$. Let $W$ be the vector space $C[a, b]$ of all continuous functions on $[a, b]$, and let $D : V \to W$ be the transformation that sends $f \mapsto f'$. Show that $D$ is a linear transformation and compute its kernel and range.*