What is on today

1 Linearly independent sets; bases

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Lay–Lay–McDonald §4.3 pp. 210 – 215

Today we study the subsets that span a vector space $V$ or a subspace $H$ as “efficiently” as possible. The main idea is that of linear independence.

An indexed set of vectors $\{v_1, \ldots, v_p\}$ in $V$ is said to be linearly independent if the equation

$$c_1v_1 + \cdots + c_pv_p = 0$$

has only the trivial solution $c_1 = \cdots = c_p = 0$. The set $\{v_1, \ldots, v_p\}$ is said to be linearly dependent if (1) has a nontrivial solution: that is, if there are weights $c_1, \ldots, c_p$ not all zero such that (1) holds. In this case, there is said to be a linear dependence relation among $v_1, \ldots, v_p$. All of this should sound familiar – we discussed the analogous definition over $V = \mathbb{R}^n$. In fact, the following theorem we saw over $\mathbb{R}^n$ also holds true:

**Theorem 1.** An indexed set $\{v_1, \ldots, v_p\}$ of two or more vectors, with $v_1 \neq 0$, is linearly dependent if and only if some $v_j$ (with $j > 1$) is a linear combination of the preceding vectors $v_1, \ldots, v_{j-1}$.

**Example 2.** Let $p_1(t) = 1, p_2(t) = t, p_3(t) = 4 - t$. Is the set $\{p_1, p_2, p_3\}$ linearly independent in the vector space of polynomials of degree at most 1?

**Definition 3.** Let $H$ be a subspace of a vector space $V$. An indexed set of vectors $B = \{b_1, \ldots, b_p\}$ in $V$ is a basis for $H$ if

1. $B$ is a linearly independent set, and
2. the subspace spanned by $B$ coincides with $H$; that is, $H = \text{Span}\{b_1, \ldots, b_p\}$.

**Example 4.** Let $A$ be an invertible $n \times n$ matrix. Then the columns of $A$ form a basis for $\mathbb{R}^n$ because they are linearly independent and they span $\mathbb{R}^n$, by the Invertible Matrix Theorem.
Example 5. Let $e_1, \ldots, e_n$ be the columns of the $n \times n$ identity matrix $I_n$:

$$
e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

The set $\{e_1, \ldots, e_n\}$ is called the standard basis of $\mathbb{R}^n$.

Example 6. Let $v_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, v_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, v_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. Determine if $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.

Example 7. Let $S = \{1, t, \ldots, t^n\}$. Verify that $S$ is a basis for $P_n$. This is called the standard basis for $P_n$.

We’ll see, a basis is an “efficient” spanning set that contains no unnecessary vectors. In fact, a basis can be constructed from a spanning set by discarding unneeded vectors.
Example 8. Let \( \mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix} \), and \( H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \). Note that \( \mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2 \) and show that \( \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} \). Then find a basis for \( H \).

The next theorem generalizes the previous example:

**Theorem 9** (Spanning set theorem). Let \( S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \) be a set in \( V \), and let \( H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \).

1. If one of the vectors in \( S \) – say \( \mathbf{v}_k \) – is a linear combination of the remaining vectors in \( S \), then the set formed from \( S \) by removing \( \mathbf{v}_k \) still spans \( H \).

2. If \( H \neq \{\mathbf{0}\} \), some subset of \( S \) is a basis for \( H \).

We know how to find vectors that span the null space of a matrix \( A \) (compute reduced echelon form, write the basic variables in terms of free variables, and decompose as a linear combination of vectors using the free variables as weights); in fact, the method produces a linearly independent set when \( \text{Nul} \ A \) contains nonzero vectors, and in that case, a basis for \( \text{Nul} \ A \). Now we describe how to find a basis for the column space, through two examples:

Example 10. Find a basis for \( \text{Col} \ B \), where \( B = [b_1 \ b_2 \ \cdots \ b_5] = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \).
Example 11. *It can be shown that the matrix* $A = [a_1 \ a_2 \ \cdots \ a_5] = \begin{bmatrix}
1 & 4 & 0 & 2 & -1 \\
3 & 12 & 1 & 5 & 5 \\
2 & 8 & 1 & 3 & 2 \\
5 & 20 & 2 & 8 & 2 \\
\end{bmatrix}$

*is row equivalent to the matrix* $B$ *in the previous example. Find a basis for* $\text{Col} \ A$.

These two examples illustrate the following useful fact:

**Theorem 12.** The pivot columns of a matrix $A$ form a basis for $\text{Col} \ A$. 