Professor Jennifer Balakrishnan, *jbala@bu.edu*

What is on today

1 Linearly independent sets; bases

Lay–Lay–McDonald §4.3 pp. 210 – 215

Today we study the subsets that span a vector space V or a subspace H as "efficiently" as possible. The main idea is that of linear independence.

An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in V is said to be *linearly independent* if the equation

$$c_1\mathbf{v}_1 + \cdots c_p\mathbf{v}_p = \mathbf{0} \tag{1}$$

has only the trivial solution $c_1 = \cdots = c_p = 0$. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be *linearly dependent* if (1) has a nontrivial solution: that is, if there are weights $c_1, \dots, c_p$ not all zero such that (1) holds. In this case, there is said to be a *linear dependence relation* among $\mathbf{v}_1, \dots, \mathbf{v}_p$. All of this should sound familiar – we discussed the analogous definition over $V = \mathbb{R}^n$. In fact, the following theorem we saw over $\mathbb{R}^n$ also holds true:

Theorem 1. *An indexed set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors $\mathbf{v}_1, \dots, \mathbf{v}_j$.*

Example 2. *Let $\mathbf{p}_1(t) = 1, \mathbf{p}_2(t) = t, \mathbf{p}_3(t) = 4 - t$. Is the set $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ linearly independent in the vector space of polynomials of degree at most 1?*

Definition 3. *Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a basis for H if*

1. *$\mathcal{B}$ is a linearly independent set, and*
2. *the subspace spanned by $\mathcal{B}$ coincides with H; that is, $H = \mathrm{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$.*

Example 4. *Let A be an invertible $n \times n$ matrix. Then the columns of A form a basis for $\mathbb{R}^n$ because they are linearly independent and they span $\mathbb{R}^n$, by the Invertible Matrix Theorem.*

Example 5. *Let* $\mathbf{e}_1, \ldots, \mathbf{e}_n$ *be the columns of the* $n \times n$ *identity matrix* I_n*:*

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ *is called the standard basis of* $\mathbb{R}^n$*.*

Example 6. *Let* $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$*. Determine if* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *is a basis for* $\mathbb{R}^3$*.*

Example 7. *Let* $S = \{1, t, \ldots, t^n\}$*. Verify that* S *is a basis for* P_n*. This is called the standard basis for* P_n*.*

We we'll see, a basis is an "efficient" spanning set that contains no unnecessary vectors. In fact, a basis can be constructed from a spanning set by discarding unneeded vectors.

Example 8. *Let* $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$, *and* $H = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. *Note that* $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$ *and show that Span* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. *Then find a basis for* H.

The next theorem generalizes the previous example:

Theorem 9 (Spanning set theorem)**.** *Let* $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ *be a set in* V, *and let* $H = \operatorname{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

1. *If one of the vectors in* S *– say* $\mathbf{v}_k$ *– is a linear combination of the remaining vectors in* S, *then the set formed from* S *by removing* $\mathbf{v}_k$ *still spans* H.
2. *If* $H \neq \{\mathbf{0}\}$, *some subset of* S *is a basis for* H.

We know how to find vectors that span the null space of a matrix A (compute reduced echelon form, write the basic variables in terms of free variables, and decompose as a linear combination of vectors using the free variables as weights); in fact, the method produces a linearly independent set when $\operatorname{Nul} A$ contains nonzero vectors, and in that case, a *basis* for $\operatorname{Nul} A$. Now we describe how to find a basis for the column space, through two examples:

Example 10. *Find a basis for* $\operatorname{Col} B$, *where* $B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_5] = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Example 11. *It can be shown that the matrix* $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_5] = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 2 \end{bmatrix}$ *is row equivalent to the matrix* B *in the previous example. Find a basis for* $\operatorname{Col} A$.

These two examples illustrate the following useful fact:

Theorem 12. *The pivot columns of a matrix* A *form a basis for* $\operatorname{Col} A$.