MA 242 (Linear Algebra)   Lecture 14: March 23, 2021   Section C1

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What is on today

1 Linearly independent sets; bases

Lay–Lay–McDonald §4.3 pp. 210 – 215

Today we study the subsets that span a vector space V or a subspace H as "efficiently" as possible. The main idea is that of linear independence.

An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in V is said to be *linearly independent* if the equation

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{1}$$

has only the trivial solution $c_1 = \cdots = c_p = 0$. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be *linearly dependent* if (1) has a nontrivial solution: that is, if there are weights $c_1, \dots, c_p$ not all zero such that (1) holds. In this case, there is said to be a *linear dependence relation* among $\mathbf{v}_1, \dots, \mathbf{v}_p$. All of this should sound familiar – we discussed the analogous definition over $V = \mathbb{R}^n$. In fact, the following theorem we saw over $\mathbb{R}^n$ also holds true:

Theorem 1. *An indexed set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors $\mathbf{v}_1, \dots, \mathbf{v}_j$.*

Example 2. *Let $\mathbf{p}_1(t) = 1, \mathbf{p}_2(t) = t, \mathbf{p}_3(t) = 4 - t$. Is the set $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ linearly independent in the vector space of polynomials of degree at most 1?*

can we write p_3 in terms of p_1 and p_2?

$p_3(t) = 4 - t = \underline{4} \cdot p_1(t) \underline{-} p_2(t)$ so the set $\{p_1, p_2, p_3\}$ is NOT linearly independent.

Definition 3. *Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a basis for H if*

1. *$\mathcal{B}$ is a linearly independent set, and*
2. *the subspace spanned by $\mathcal{B}$ coincides with H; that is, $H = \operatorname{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$.*

says every elt in H can be written as a lin. comb. of $b_1, \dots, b_p$.

Example 4. *Let A be an invertible $n \times n$ matrix. Then the columns of A form a basis for $\mathbb{R}^n$ because they are linearly independent and they span $\mathbb{R}^n$, by the Invertible Matrix Theorem.*

Example 5. *Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the columns of the $n \times n$ identity matrix I_n:*

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

I has det 1, so it's invertible, so by Ex 4, columns form basis for $\mathbb{R}^n$

The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is called the standard basis of $\mathbb{R}^n$.

Example 6. *Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.*

Can do this a few different ways:
- compute echelon form of matrix $A := [v_1\ v_2\ v_3]$ & count pivots
- compute solutions to $Ax=0$ & check if it's just trivial solution
- compute det A, check if it's 0 or not.

$$\det A: \begin{vmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 \\ 7 & 5 \end{vmatrix} - 6 \begin{vmatrix} -4 & -2 \\ 1 & 1 \end{vmatrix}$$

$$= 3(5-7) - 6(-4+2)$$

$$= 3(-2) - 6(-2) = -6 + 12 = 6 \neq 0$$

$\Rightarrow$ A is invertible & $\{v_1, v_2, v_3\}$ is a basis!

Example 7. *Let $S = \{1, t, \ldots, t^n\}$. Verify that S is a basis for P_n. This is called the standard basis for P_n.*

polynomials of deg $\leq n$

basis = linear independence + span.

poly of deg $\leq n$ looks like

$c_0 + c_1 t + \cdots + c_n t^n = \sum_{i=0}^{n} c_i t^i$ for $t^i \in \{1, t, \ldots, t^n\} =: S$

Suppose

$c_0 + c_1 t + \cdots + c_n t^n = 0$

if nonzero, this will be 0 at most n times

this equality is not satisfied unless LHS = 0

this is the 0 polynomial. literally always 0

$\Rightarrow$, we need LHS to be 0 polynomial as well $\Rightarrow$ all $c_i = 0$ $\Rightarrow$ lin indep.

Alternatively: translate $c_0 + c_1 t + \cdots + c_n t^n \to (c_0, c_1, \ldots, c_n)$ vector of length $n+1$.

$1 \to (1, 0, 0, \ldots, 0)$

$t \to (0, 1, 0, \ldots, 0)$

$t^2 \to (0, 0, 1, 0, \ldots, 0)$ $\cdots$ $t^n \to (0, \ldots, 0, 1)$

We we'll see, a basis is an "efficient" spanning set that contains no unnecessary vectors. In fact, a basis can be constructed from a spanning set by discarding unneeded vectors.

e.g. for P_2 : polys of deg at most 2

$= \{c_0 + c_1 t + c_2 t^2\}$

(c_0, c_1, c_2)

$1 \mapsto (1, 0, 0)$

$t \mapsto (0, 1, 0)$

$t^2 \mapsto (0, 0, 1)$

is a basis for $\mathbb{R}^3$

$\{1, t, t^2\}$ basis for P_2

Example 8. *Let* $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$, *and* $H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. *Note that* $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$ *and show that Span* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. *Then find a basis for* H.

Why is Span $\{v_1, v_2, v_3\}$ = Span $\{v_1, v_2\}$?

$\mathrm{Span}\{v_1, v_2, \underbrace{5v_1+3v_2}_{=5\cdot v_1 + 3\cdot v_2}\} = \mathrm{Span}\{v_1, v_2\}$ ✓

What is a basis for H? check lin. indep. of v_1, v_2 : they aren't scalar multiples of each other, so lin. indep. So basis for H is $\{v_1, v_2\}$.

The next theorem generalizes the previous example:

Theorem 9 (Spanning set theorem)**.** *Let* $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ *be a set in* V, *and let* $H = \mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

1. *If one of the vectors in* S *– say* $\mathbf{v}_k$ *– is a linear combination of the remaining vectors in* S, *then the set formed from* S *by removing* $\mathbf{v}_k$ *still spans* H.
2. *If* $H \neq \{\mathbf{0}\}$, *some subset of* S *is a basis for* H.

We know how to find vectors that span the null space of a matrix A (compute reduced echelon form, write the basic variables in terms of free variables, and decompose as a linear combination of vectors using the free variables as weights); in fact, the method produces a linearly independent set when $\mathrm{Nul}\, A$ contains nonzero vectors, and in that case, a *basis* for $\mathrm{Nul}\, A$. Now we describe how to find a basis for the column space, through two examples:

Example 10. *Find a basis for* $\mathrm{Col}\, B$, *where* $B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_5] = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

See that $b_2 = 4b_1$

$b_4 = 2\cdot b_1 - 1\cdot b_3$

See that $\{b_1, b_3, b_5\}$ is a lin. indep set & span ⇒ basis

also $\{b_2, b_3, b_5\}$ is a lin. indep set & span ⇒ basis

Example 11. *It can be shown that the matrix* $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_5] = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 2 \end{bmatrix}$ *is row equivalent to the matrix B in the previous example. Find a basis for* Col A.

B: $b_2 = 4b_1$, $b_4 = 2b_1 - b_3$ $\Rightarrow$ $a_2 \overset{?}{=} 4a_1$ ✓, $a_4 \overset{?}{=} 2a_1 - a_3$ ✓ (by row equivalence)

basis for Col A = $\{a_1, a_3, a_5\}$.

These two examples illustrate the following useful fact:

Theorem 12. *The pivot columns of a matrix A form a basis for* Col A.

$\mathbb{R}^2$: basis is $\{(1,0), (0,1)\}$

another basis is $\{(2,0), (0,2)\}$ $\quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$\vdots$

$\{(1,1), (0,1)\}$ $\quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$\vdots$ $\longleftrightarrow$ cols of inv. 2×2 matrices

$(x,y) = x\cdot(1,0) + y(0,1) = \frac{x}{2}(2,0) + \frac{y}{2}(0,2) = \ldots$

but e.g. $\{(1,1), (2,2)\}$ not a basis for $\mathbb{R}^2$