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What is on today

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1 Coordinate systems

Lay–Lay–McDonald §4.4 pp. 218 – 224

An important reason for specifying a basis \mathcal{B} for a vector space V is to impose a “coordinate system” on V . This section will show that if \mathcal{B} contain n vectors, then the coordinate system will make V act like \mathbb{R}^n .

Theorem 1 (Unique representation). *Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V . Then for each $\mathbf{x} \in V$, there exists a unique set of scalars c_1, \dots, c_n such that $\mathbf{x} = c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n$.*

Definition 2. *Suppose $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x} \in V$. The coordinates of \mathbf{x} relative to \mathcal{B} (“ \mathcal{B} -coordinates of \mathbf{x} ”) are the weights c_1, \dots, c_n such that $\mathbf{x} = c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n$.*

If c_1, \dots, c_n are the \mathcal{B} -coordinates of \mathbf{x} , then the vector in \mathbb{R}^n given by $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is the coordinate vector x relative to \mathcal{B} . The mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the coordinate mapping determined by \mathcal{B} .

Example 3. *Consider a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ for \mathbb{R}^2 where $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Suppose $\mathbf{x} \in \mathbb{R}^2$ has the coordinate vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find \mathbf{x} .*

Example 4. *The entries in the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ are the coordinates of \mathbf{x} relative to the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$, since $\begin{bmatrix} 1 \\ 6 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1\mathbf{e}_1 + 6\mathbf{e}_2$.*

Example 5. Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of \mathbf{x} relative to \mathcal{B} .

Suppose we have a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$. Let $P_{\mathcal{B}} = [\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_n]$. The vector equation

$$\mathbf{x} = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 + \cdots + c_n \mathbf{b}_n$$

is equivalent to

$$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}.$$

We call $P_{\mathcal{B}}$ the *change of coordinates matrix* from \mathcal{B} to the standard basis in \mathbb{R}^n .

Since the columns of $P_{\mathcal{B}}$ form a basis for \mathbb{R}^n , we have that $P_{\mathcal{B}}$ is invertible, and we have

$$P_{\mathcal{B}}^{-1} \mathbf{x} = [\mathbf{x}]_{\mathcal{B}},$$

which tells us that $P_{\mathcal{B}}^{-1}$ gives the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$!

Theorem 6. Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V . Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto \mathbb{R}^n .

The above coordinate mapping is an important example of an isomorphism from V to \mathbb{R}^n . In general, a one-to-one linear transformation from a vector space V onto a vector space W is called an isomorphism from V onto W .

Example 7. Let $\mathcal{B} = \{1, t, t^2, t^3\}$ be the standard basis of the space P_3 of polynomials. A

typical element \mathbf{p} of P_3 has the form $\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$. We have that $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$.

The coordinate mapping $\mathbf{p} \mapsto [\mathbf{p}]_{\mathcal{B}}$ is an isomorphism from P_3 to \mathbb{R}^4 , and all vector space operations in P_3 correspond to operations in \mathbb{R}^4 .

Example 8. Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$ and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then \mathcal{B} is a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Determine if $\mathbf{x} \in H$ and if it is, find the coordinate vector of \mathbf{x} relative to \mathcal{B} .

2 The dimension of a vector space

Lay–Lay–McDonald §4.5 pp. 227 – 230

Earlier, we saw that a vector space V with a basis \mathcal{B} containing n vectors is isomorphic to \mathbb{R}^n . Today we show that this number n is an intrinsic property (the *dimension*) of the space V that does not depend on the choice of basis.

Theorem 9. *If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.*

The previous theorem implies that if a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, then each linearly independent set in V has no more than n vectors.

Theorem 10. *If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.*

Proof. Let \mathcal{B}_1 be a basis of n vectors, and \mathcal{B}_2 be any other basis of V . Since \mathcal{B}_1 is a basis and \mathcal{B}_2 is linearly independent, \mathcal{B}_2 has no more than n vectors, by the previous theorem. Also, since \mathcal{B}_2 is a basis and \mathcal{B}_1 is linearly independent, \mathcal{B}_2 has at least n vectors. Thus, \mathcal{B}_2 consists of exactly n vectors. \square

This leads us to the following definition:

Definition 11. *If V is spanned by a finite set, then V is said to be finite-dimensional, and the dimension of V , written as $\dim V$, is the number of vectors in a basis for V . The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be infinite-dimensional.*

Example 12. *What is $\dim \mathbb{R}^n$? What about $\dim P_2$, where P_2 denotes the vector space of polynomials of degree at most 2?*

Example 13. Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

The next theorem serves as a natural counterpart to the Spanning Set Theorem:

Theorem 14. Let H be a subspace of a finite-dimensional vector space V . Any linearly independent set in H can be expanded, if necessary, to a basis for H . Also, H is finite-dimensional and $\dim H \leq \dim V$.

When the dimension of a vector space (or subspace) is known, the search for a basis is simplified by the next theorem. It says that if a set has the right number of elements, then one has only to show that the set is linearly independent or that it spans the space. This is important in a number of applications, where linear independence is easier to check than spanning.

Theorem 15 (The basis theorem). Let V be a p -dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V . Any set of exactly p elements that spans V is automatically a basis for V .

Now we apply the notion of dimension to two familiar vector subspaces: the null space and column space. We have the following:

Theorem 16. Let A be an $m \times n$ matrix. The dimension of $\text{Nul } A$ is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$, and the dimension of $\text{Col } A$ is the number of pivot columns in A .

Example 17. *Find the dimensions of the null space and the column space of*

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$