MA 242 (Linear Algebra)   Lecture 15: March 25, 2021   Section C1

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What is on today

1 Coordinate systems

Lay–Lay–McDonald §4.4 pp. 218 – 224

An important reason for specifying a basis $\mathcal{B}$ for a vector space V is to impose a "coordinate system" on V. This section will show that if $\mathcal{B}$ contain n vectors, then the coordinate system will make V act like $\mathbb{R}^n$.

Theorem 1 (Unique representation)**.** *Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x} \in V$, there exists a unique set of scalars $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.*

Definition 2. *Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x} \in V$. The coordinates of $\mathbf{x}$ relative to $\mathcal{B}$ ("$\mathcal{B}$-coordinates of $\mathbf{x}$") are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n$.*

If $c_1, \ldots, c_n$ are the $\mathcal{B}$-coordinates of $\mathbf{x}$, then the vector in $\mathbb{R}^n$ given by $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is the *coordinate vector x relative to $\mathcal{B}$*. The mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the *coordinate mapping determined by $\mathcal{B}$*.

Example 3. *Consider a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ for $\mathbb{R}^2$ where $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Suppose $\mathbf{x} \in \mathbb{R}^2$ has the coordinate vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find $\mathbf{x}$.*

span $\mathbb{R}^2$ ✓ and lin. indep. ✓

$$x = -2b_1 + 3b_2$$
$$= -2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}.$$

Example 4. *The entries in the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ are the coordinates of $\mathbf{x}$ relative to the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$, since* $\begin{bmatrix} 1 \\ 6 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1\mathbf{e}_1 + 6\mathbf{e}_2$.

$\rightarrow P_2$

$p(t) = a_0 + a_1 t + a_2 t^2$

$\updownarrow$

$(a_0, a_1, a_2) \leftarrow$

$t^1 \quad t^2$

Example 5. *Let* $\mathbf{b}_1 = \begin{bmatrix}2\\1\end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix}-1\\1\end{bmatrix}, \mathbf{x} = \begin{bmatrix}4\\5\end{bmatrix}$*, and* $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$*. Find the coordinate vector* $[\mathbf{x}]_{\mathcal{B}}$ *of* $\mathbf{x}$ *relative to* $\mathcal{B}$*.*

$\to \begin{pmatrix}2&-1\\1&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}4\\5\end{pmatrix} \Rightarrow \begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}2&-1\\1&1\end{pmatrix}^{-1}\begin{pmatrix}4\\5\end{pmatrix}$

$[x]_{\mathcal{B}} = \begin{bmatrix}c_1\\c_2\end{bmatrix} = c_1 b_1 + c_2 b_2 = \boxed{c_1\begin{pmatrix}2\\1\end{pmatrix}+c_2\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}4\\5\end{pmatrix}}$. So want to solve for c_1, c_2.

$$\begin{pmatrix}2&-1&4\\1&1&5\end{pmatrix}\sim\begin{pmatrix}1&1&5\\2&-1&4\end{pmatrix}\overset{-2\cdot\#1+\#2}{\sim}\begin{pmatrix}1&1&5\\0&-3&-6\end{pmatrix}\underset{\div -3}{\sim}\begin{pmatrix}1&1&5\\0&1&2\end{pmatrix}\sim\begin{pmatrix}1&0&3\\0&1&2\end{pmatrix}$$

$\Rightarrow c_1=3,\ c_2=2 \Rightarrow [x]_{\mathcal{B}} = \begin{bmatrix}3\\2\end{bmatrix}$. $\boxed{\text{Claim: } 3\begin{pmatrix}2\\1\end{pmatrix}+2\begin{pmatrix}-1\\1\end{pmatrix} = 4\begin{pmatrix}1\\0\end{pmatrix}+5\begin{pmatrix}0\\1\end{pmatrix}.}$

Suppose we have a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$. Let $P_{\mathcal{B}} = [\mathbf{b}_1 \ \ \mathbf{b}_2 \ \ \cdots \ \ \mathbf{b}_n]$. The vector equation

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

is equivalent to

$$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}.$$

e.g. $\mathcal{B} = \left\{\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right\}$

$P_{\mathcal{B}} = \begin{pmatrix}2&-1\\1&1\end{pmatrix}$

$\begin{pmatrix}4\\5\end{pmatrix} = x = \begin{pmatrix}2&-1\\1&1\end{pmatrix}[x]_{\mathcal{B}}$

$\Rightarrow [x]_{\mathcal{B}} = \begin{pmatrix}2&-1\\1&1\end{pmatrix}^{-1}\cdot\begin{pmatrix}4\\5\end{pmatrix}$

We call $P_{\mathcal{B}}$ the *change of coordinates matrix* from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.

Since the columns of $P_{\mathcal{B}}$ form a basis for $\mathbb{R}^n$, we have that $P_{\mathcal{B}}$ is invertible, and we have

$$P_{\mathcal{B}}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}},$$

which tells us that $P_{\mathcal{B}}^{-1}$ gives the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$!

Theorem 6. *Let* $\mathcal{B} = \{\mathbf{b}_1, \cdots \mathbf{b}_n\}$ *be a basis for a vector space* V*. Then the coordinate mapping* $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ *is a one-to-one linear transformation from* V *onto* $\mathbb{R}^n$*.*

The above coordinate mapping is an important example of an isomorphism from V to $\mathbb{R}^n$. In general, a one-to-one linear transformation from a vector space V onto a vector space W is called an isomorphism from V onto W.

Example 7. *Let* $\mathcal{B} = \{1, t, t^2, t^3\}$ *be the standard basis of the space* P_3 *of polynomials. A typical element* $\mathbf{p}$ *of* P_3 *has the form* $\mathbf{p}(t) = a_0 + a_1t + a_2t^2 + a_3t^3$*. We have that* $[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix}a_0\\a_1\\a_2\\a_3\end{bmatrix}$*.*

The coordinate mapping $\mathbf{p} \mapsto [\mathbf{p}]_{\mathcal{B}}$ *is an isomorphism from* P_3 *to* $\mathbb{R}^4$*, and all vector space operations in* P_3 *correspond to operations in* $\mathbb{R}^4$*.*

Example 8. *Let* $\mathbf{v}_1 = \begin{bmatrix}3\\6\\2\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix}, \mathbf{x} = \begin{bmatrix}3\\12\\7\end{bmatrix}$ *and* $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$*. Then* $\mathcal{B}$ *is a basis for* $H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$*. Determine if* $\mathbf{x} \in H$ *and if it is, find the coordinate vector of* $\mathbf{x}$ *relative to* $\mathcal{B}$*.*

Are there c_1, c_2 s.t. $c_1v_1 + c_2v_2 = x$?

$c_1\begin{pmatrix}3\\6\\2\end{pmatrix}+c_2\begin{pmatrix}-1\\0\\1\end{pmatrix}=\begin{pmatrix}3\\12\\7\end{pmatrix}$

Check if system is consistent:

$$\begin{pmatrix}3&-1&3\\6&0&12\\2&1&7\end{pmatrix}\underset{\div 6}{\sim}\begin{pmatrix}3&-1&3\\1&0&2\\2&1&7\end{pmatrix}$$

$$\sim\begin{pmatrix}1&0&2\\2&1&7\\3&-1&3\end{pmatrix}\underset{\substack{-2\cdot\#1+\#2\\-3\cdot\#1+\#3}}{\sim}\begin{pmatrix}1&0&2\\0&-1&-3\\0&1&3\end{pmatrix}\sim\begin{pmatrix}1&0&2\\0&1&3\\0&0&0\end{pmatrix}$$

$\Rightarrow c_1 = 2,\ c_2 = 3 \Rightarrow x \in H$, $x = 2\cdot v_1 + 3\cdot v_2 = \begin{bmatrix}2\\3\end{bmatrix}_{\mathcal{B}}$

2 The dimension of a vector space

Lay–Lay–McDonald §4.5 pp. 227 – 230

Earlier, we saw that a vector space V with a basis $\mathcal{B}$ containing n vectors is isomorphic to $\mathbb{R}^n$. Today we show that this number n is an intrinsic property (the *dimension*) of the space V that does not depend on the choice of basis.

Theorem 9. *If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.*

The previous theorem implies that if a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, then each linearly independent set in V has no more than n vectors.

Theorem 10. *If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.*

Proof. Let $\mathcal{B}_1$ be a basis of n vectors, and $\mathcal{B}_2$ be any other basis of V. Since $\mathcal{B}_1$ is a basis and $\mathcal{B}_2$ is linearly independent, $\mathcal{B}_2$ has no more than n vectors, by the previous theorem. Also, since $\mathcal{B}_2$ is a basis and $\mathcal{B}_1$ is linearly independent, $\mathcal{B}_2$ has at least n vectors. Thus, $\mathcal{B}_2$ consists of exactly n vectors. □

This leads us to the following definition:

Definition 11. *If V is spanned by a finite set, then V is said to be finite-dimensional, and the dimension of V, written as $\dim V$, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be infinite-dimensional.*

Example 12. *What is $\dim \mathbb{R}^n$? What about $\dim P_2$, where P_2 denotes the vector space of polynomials of degree at most 2?*

$\dim \mathbb{R}^n = n$

$\dim P_2 = 3$ (recall: a basis for P_2 is $\{1, t, t^2\}$)

Example 13. *Find the dimension of the subspace*

at most 4

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

$$= \begin{pmatrix} a \\ 5a \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -3b \\ 0 \\ b \\ 0 \end{pmatrix} + \begin{pmatrix} 6c \\ 0 \\ -2c \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 4d \\ -d \\ 5d \end{pmatrix}$$

$$= a\begin{pmatrix} 1 \\ 5 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 6 \\ 0 \\ -2 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 4 \\ -1 \\ 5 \end{pmatrix}$$

$$= \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 5 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ -1 \\ 5 \end{pmatrix} \right\}$$

$$= \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 5 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ -1 \\ 5 \end{pmatrix} \right\}$$

Now check if these 3 vectors are lin independent (they are) $\Rightarrow$ dim $H = 3$.

The next theorem serves as a natural counterpart to the Spanning Set Theorem:

Theorem 14. *Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and* $\dim H \leq \dim V$*.*

When the dimension of a vector space (or subspace) is known, the search for a basis is simplified by the next theorem. It says that if a set has the right number of elements, then one has only to show that the set is linearly independent or that it spans the space. This is important in a number of applications, where linear independence is easier to check than spanning.

Theorem 15 (The basis theorem)**.** *Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.*

Now we apply the notion of dimension to two familiar vector subspaces: the null space and column space. We have the following:

Theorem 16. *Let A be an $m \times n$ matrix. The dimension of* $\mathrm{Nul}\, A$ *is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$, and the dimension of* $\mathrm{Col}\, A$ *is the number of pivot columns in A.*

Example 17. *Find the dimensions of the null space and the column space of*

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

$$\begin{pmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \\ -3 & 6 & -1 & 1 & -7 & 0 \end{pmatrix} \underset{3\#1+\#3}{\overset{-2\cdot\#1+\#2}{\sim}} \begin{pmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 5 & 10 & -10 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\uparrow$ x_2 $\uparrow$ x_4 x_5

pivot columns

2 pivot columns $\to$ dim Col $A = 2$

3 free vars $\to$ dim Nul $A = 3$.