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What is on today

1 Rank

Lay–Lay–McDonald §4.6 pp. 235 – 237

If A is an $m \times n$ matrix, each row of A has n entries and so can be identified with a vector in $\mathbb{R}^n$. The set of all linear combinations of the row vectors is called the row space of A and is denoted $\operatorname{Row} A$. Each row has n entries, so $\operatorname{Row} A$ is a subspace of $\mathbb{R}^n$. Since the rows of A are identified with the columns of A^T, we could equivalently write $\operatorname{Col} A^T$ in place of $\operatorname{Row} A$.

Theorem 1. *If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.*

Now recall this example from the last class:

Example 2. *Find the dimensions of the null space and the column space of*

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

We define the *rank* of a matrix A to be the dimension of the column space of A. The following is a nice result about how the rank and the dimension of the null space are related:

Theorem 3. *Let A be an $m \times n$ matrix. We have*

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n.$$

To summarize, here is a collection of things we've learned over the last few classes and how they relate to invertibility of a matrix:

Theorem 4 (Invertible Matrix Theorem (continued))**.** *Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix:*

1. *The columns of A form a basis of $\mathbb{R}^n$.*
2. $\operatorname{Col} A = \mathbb{R}^n$.
3. $\dim \operatorname{Col} A = n$.
4. $\operatorname{rank} A = n$.
5. $\operatorname{Nul} A = \{\mathbf{0}\}$.
6. $\dim \operatorname{Nul} A = 0$.

2 Change of basis

Lay–Lay–McDonald §4.7 pp. 241 – 244

In some applications, a problem is described initially using a basis $\mathcal{B}$, but the problem is easier to solve by changing $\mathcal{B}$ to a new basis $\mathcal{C}$. Each vector is assigned a new $\mathcal{C}$-coordinate vector. In this section, we study how $[\mathbf{x}]_\mathcal{C}$ and $[\mathbf{x}]_\mathcal{B}$ are related for each $\mathbf{x} \in V$.

Example 5. *Consider two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ for a vector space V, such that $\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2$ and $\mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$. Suppose $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$. That is, suppose $[\mathbf{x}]_\mathcal{B} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[\mathbf{x}]_\mathcal{C}$.*

We can generalize the argument in the example above to produce the following theorem:

Theorem 6. *Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that*

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}}.$$

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = [[\mathbf{b}_1]_{\mathcal{C}} \quad [\mathbf{b}_2]_{\mathcal{C}} \quad \cdots \quad [\mathbf{b}_n]_{\mathcal{C}}].$$

The matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is called the *change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$*. Multiplication by $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ converts $\mathcal{B}$-coordinates into $\mathcal{C}$-coordinates. The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are linearly independent because they are the coordinate vectors of the linearly independent set $\mathcal{B}$, and since $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is square, it is invertible by the Invertible Matrix Theorem. Indeed, we have

$$\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}[\mathbf{x}]_{\mathcal{C}} = [\mathbf{x}]_{\mathcal{B}}.$$

Thus $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}$ is the matrix that converts $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates. That is,

$$\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}.$$

Now if $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{E}$ is the *standard basis* $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ in $\mathbb{R}^n$, then $[\mathbf{b}_1]_{\mathcal{E}} = \mathbf{b}_1$ and likewise for the other vectors in the basis. In this case, $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ is the same as the change-of-coordinates matrix $P_{\mathcal{B}}$ introduced previously, namely,

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n].$$

Example 7. *Let $\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, and consider the bases for $\mathbb{R}^2$ given by $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.*

Example 8. *Let* $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$*, and consider the bases for* $\mathbb{R}^2$ *given by* $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ *and* $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$.

1. *Find the change-of-coordinates matrix from* $\mathcal{C}$ *to* $\mathcal{B}$.

2. *Find the change-of-coordinates matrix from* $\mathcal{B}$ *to* $\mathcal{C}$.