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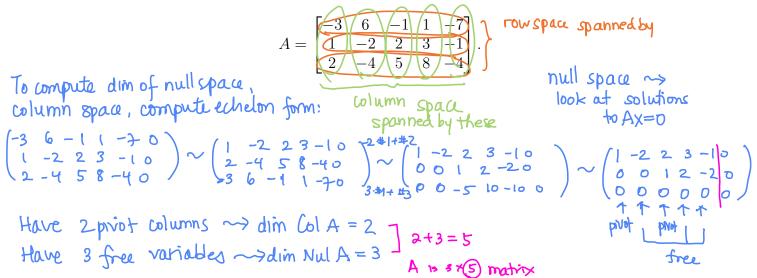
What is on today

1   Rank

Lay–Lay–McDonald §4.6 pp. 235 – 237

If A is an $m \times n$ matrix, each row of A has n entries and so can be identified with a vector in $\mathbb{R}^n$. The set of all linear combinations of the row vectors is called the row space of A and is denoted Row A. Each row has n entries, so Row A is a subspace of $\mathbb{R}^n$. Since the rows of A are identified with the columns of A^T, we could equivalently write Col A^T in place of Row A.

Theorem 1. *If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.*

Now recall this example from the last class:

Example 2. *Find the dimensions of the null space and the column space of*

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

row space spanned by

column space spanned by these

null space ⟿ look at solutions to Ax=0

To compute dim of null space, column space, compute echelon form:

$$\begin{pmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \\ -3 & 6 & -1 & 1 & -7 & 0 \end{pmatrix} \overset{-2\cdot\#1+\#2}{\underset{3\cdot\#1+\#3}{\sim}} \begin{pmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & -5 & 10 & -10 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot   pivot   free

Have 2 pivot columns ⟿ dim Col A = 2
Have 3 free variables ⟿ dim Nul A = 3   } 2+3=5

A is 3×(5) matrix

We define the *rank* of a matrix A to be the dimension of the column space of A. The following is a nice result about how the rank and the dimension of the null space are related:

Theorem 3. *Let A be an $m \times n$ matrix. We have*

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n.$$

To summarize, here is a collection of things we've learned over the last few classes and how they relate to invertibility of a matrix:

Theorem 4 (Invertible Matrix Theorem (continued)). *Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix:*

1. *The columns of A form a basis of $\mathbb{R}^n$.*

2. $\operatorname{Col} A = \mathbb{R}^n$.

3. $\dim \operatorname{Col} A = n$.

4. $\operatorname{rank} A = n$.

5. $\operatorname{Nul} A = \{\mathbf{0}\}$.

6. $\dim \operatorname{Nul} A = 0$.

2   Change of basis

Lay–Lay–McDonald §4.7 pp. 241 – 244

In some applications, a problem is described initially using a basis $\mathcal{B}$, but the problem is easier to solve by changing $\mathcal{B}$ to a new basis $\mathcal{C}$. Each vector is assigned a new $\mathcal{C}$-coordinate vector. In this section, we study how $[\mathbf{x}]_{\mathcal{C}}$ and $[\mathbf{x}]_{\mathcal{B}}$ are related for each $\mathbf{x} \in V$.

Example 5. *Consider two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ for a vector space V, such that $\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2$ and $\mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$. Suppose $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$. That is, suppose $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[\mathbf{x}]_{\mathcal{C}}$.*

$$[x]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 3b_1 + b_2 \;;\; b_1 = 4c_1 + c_2$$

$$b_2 = -6c_1 + c_2$$

$$3(4c_1 + c_2) + (-6c_1 + c_2)$$

$$= 12c_1 + 3c_2 - 6c_1 + c_2 = 6c_1 + 4c_2$$

$$[x]_C = \underset{C \leftarrow B}{P} \cdot [x]_B$$

$$\begin{bmatrix} 6 \\ 4 \end{bmatrix} = [\,[b_1]_C \; [b_2]_C\,] \, [x]_B$$

$$\begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \checkmark$$

$$\rightsquigarrow [x]_C = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

$*$

We can generalize the argument in the example above to produce the following theorem:

Theorem 6. *Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ such that*

$$[\mathbf{x}]_\mathcal{C} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P}[\mathbf{x}]_\mathcal{B}.$$

The columns of $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = [[\mathbf{b}_1]_\mathcal{C} \quad [\mathbf{b}_2]_\mathcal{C} \quad \cdots \quad [\mathbf{b}_n]_\mathcal{C}].$$

The matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ is called the *change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$*. Multiplication by $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ converts $\mathcal{B}$-coordinates into $\mathcal{C}$-coordinates. The columns of $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ are linearly independent because they are the coordinate vectors of the linearly independent set $\mathcal{B}$, and since $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ is square, it is invertible by the Invertible Matrix Theorem. Indeed, we have

$$\left(\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}\right)^{-1}[\mathbf{x}]_\mathcal{C} = [\mathbf{x}]_\mathcal{B}.$$

Thus $\left(\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}\right)^{-1}$ is the matrix that converts $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates. That is,

$$\left(\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B} \leftarrow \mathcal{C}}{P}.$$

Now if $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{E}$ is the *standard basis* $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ in $\mathbb{R}^n$, then $[\mathbf{b}_1]_\mathcal{E} = \mathbf{b}_1$ and likewise for the other vectors in the basis. In this case, $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$ is the same as the change-of-coordinates matrix $P_\mathcal{B}$ introduced previously, namely,

$$P_\mathcal{B} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n].$$

Example 7. *Let $\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, and consider the bases for $\mathbb{R}^2$ given by $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$. Find the* <u>*change-of-coordinates matrix from*</u> *$\mathcal{B}$ to $\mathcal{C}$.*

want: $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left[[b_1]_C \ [b_2]_C \right]$

$$b_1 = \begin{pmatrix} -9 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 3 \\ -4 & -5 \end{pmatrix}}_{\underset{\mathcal{E} \leftarrow C}{P}} \underbrace{[b_1]_C}$$

$\underbrace{}_{[b_1]_\mathcal{E}}$ want this.

need to compute ←

$$b_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix}}_{\underset{\mathcal{E} \leftarrow C}{P}} [b_2]_C$$

$\underbrace{}_{[b_2]_\mathcal{E}}$ $\underset{C \leftarrow \mathcal{E}}{}$ want this too.

★ $\underset{C \to \mathcal{E}}{P^{-1}} = \begin{pmatrix} 1 & 3 \\ -4 & -5 \end{pmatrix}^{-1} = \frac{1}{7}\begin{pmatrix} -5 & -3 \\ 4 & 1 \end{pmatrix}$

Then take $\underset{C \to \mathcal{E}}{P^{-1}} \cdot b_1 = \frac{1}{7}\begin{pmatrix} -5 & -3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} -9 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \end{pmatrix} = [b_1]_C$

$\underset{C \to \mathcal{E}}{P^{-1}} \cdot b_2 = \frac{1}{7}\begin{pmatrix} -5 & -3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} -5 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} = [b_2]_C$

Finally,

$$\underset{C \leftarrow \mathcal{B}}{P} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$

3

Example 8. *Let* $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix},$ *and consider the bases for* $\mathbb{R}^2$ *given by* $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ *and* $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}.$

1. *Find the change-of-coordinates matrix from* $\mathcal{C}$ *to* $\mathcal{B}$.

$$\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \left[[c_1]_{\mathcal{B}} \ [c_2]_{\mathcal{B}} \right]$$

compute echelon form

can get this from: $[b_1 \ b_2 \mid c_1 \ c_2] \sim [I \mid \underset{\mathcal{B} \leftarrow \mathcal{C}}{P}]$

$$\begin{bmatrix} 1 & -2 & -7 & -5 \\ -3 & 4 & 9 & 7 \end{bmatrix} \overset{3 \cdot \#1 + \#3}{\sim} \begin{bmatrix} 1 & -2 & -7 & -5 \\ 0 & -2 & -12 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -7 & -5 \\ 0 & 1 & 6 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 6 & 4 \end{bmatrix}$$

$$\text{get} \ \underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix}.$$

2. *Find the change-of-coordinates matrix from* $\mathcal{B}$ *to* $\mathcal{C}$.

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left[[b_1]_{\mathcal{C}} \ [b_2]_{\mathcal{C}} \right]$$

compute echelon form.

Can get this from: $[c_1 \ c_2 \mid b_1 \ b_2] \sim [I \mid \underset{\mathcal{C} \leftarrow \mathcal{B}}{P}]$

also : $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left(\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} \right)^{-1}$

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 4 & -3 \\ -6 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -3/2 \\ -3 & 5/2 \end{pmatrix}$$