Professor Jennifer Balakrishnan, *jbala@bu.edu*

What is on today

1 Eigenvectors and eigenvalues

Lay–Lay–McDonald §5.1 pp. 268 – 273

Although a transformation $\mathbf{x} \mapsto A\mathbf{x}$ may transform vectors in a number of directions, it often happens that there are special vectors on which the action of A is simple. Our discussion in this chapter will be about square matrices.

Example 1. *Let* $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. *Compute* $A\mathbf{u}$ *and* $A\mathbf{v}$.

Today we will study equations of the form $A\mathbf{x} = \lambda\mathbf{x}$ where special vectors are transformed by A into scalar multiples of themselves.

Definition 2. *An eigenvector of an* $n \times n$ *matrix* A *is a nonzero vector* $\mathbf{x}$ *such that* $A\mathbf{x} = \lambda\mathbf{x}$ *for some scalar* λ. *A scalar* λ *is called an eigenvalue of* A *if there is a nontrivial solution* $\mathbf{x}$ *of* $A\mathbf{x} = \lambda\mathbf{x}$. *Such a* $\mathbf{x}$ *is called an eigenvector corresponding to* λ.

Example 3. *Let* $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. *Are* $\mathbf{u}$ *and* $\mathbf{v}$ *eigenvectors of* A*?*

Example 4. *Show that 7 is an eigenvalue of the matrix* A *in the previous example, and find the corresponding eigenvectors.*

Remark 5. *Note that while row reduction can be used to find eigenvectors, it cannot be used to find eigenvalues. An echelon form of a matrix A usually does not display the eigenvalues of A.*

Note that λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0} \tag{1}$$

has a nontrivial solution. The set of all solutions of (1) is just the null space of the matrix $A - \lambda I$. So this is a *subspace* of $\mathbb{R}^n$ and is called the *eigenspace of A corresponding to λ.*

Example 6. *Let* $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. *An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.*

Here is one situation where it's easy to compute eigenvalues:

Theorem 7. *The eigenvalues of a triangular matrix are the entries on its main diagonal.*

Proof. For simplicity, consider the 3×3 case. If A is upper-triangular, then $A - \lambda I$ has the form

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \\ &= \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}. \end{aligned}$$

We have that λ is an eigenvalue of A if and only if the equation $(A-\lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, i.e., if and only if the equation has a free variable. We see that $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a free variable if and only if at least one of the entries on the diagonal of $A - \lambda I$ is zero. This happens if and only if λ is one of the entries a_{11}, a_{22}, a_{33} in A.
We leave the case of lower-triangular matrices as an exercise. □

Example 8. *Let* $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}$. *What are the eigenvalues of A? What are the eigenvalues of B?*

What does it mean for a matrix A to have an eigenvalue of 0? This happens if and only if the equation $A\mathbf{x} = 0\mathbf{x}$ has a nontrivial solution. But this is equivalent to $A\mathbf{x} = \mathbf{0}$, which has a nontrivial solution if and only if A is not invertible. *Thus 0 is an eigenvalue of A if and only if A is not invertible.*

Here is an important result about eigenvectors that we will record for later use:

Theorem 9. *If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.*

2 The characteristic equation

Lay–Lay–McDonald §5.2 pp. 276 – 278

Now we practice finding eigenvalues:

Example 10. *Find the eigenvalues of* $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

We have the following important result: *a scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation*

$$\det(A - \lambda I) = 0.$$

Example 11. *Find the characteristic equation of* $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

It can be shown that if A is an $n \times n$ matrix, then $\det(A - \lambda I)$ is a polynomial of degree n called the *characteristic polynomial* of A.

Example 12. *The characteristic polynomial of a* 6×6 *matrix is* $\lambda^6 - 4\lambda^5 - 12\lambda^4$. *Find the eigenvalues and their multiplicities.*

The next theorem presents one use of the characteristic polynomial and is helpful for iterative methods that approximate eigenvalues. We begin with some terminology. If A and B are $n \times n$ matrices, then we say that A is *similar to* B if there is an invertible matrix P such that

$$P^{-1}AP = B.$$

Writing $Q := P^{-1}$, we also have

$$Q^{-1}BQ = A.$$

So B is also similar to A, and we say that A *and* B *are similar.*

Theorem 13. *If* $n \times n$ *matrices* A *and* B *are similar, then they have the same characteristic polynomial and hence the same eigenvalues with the same multiplicities.*

Proof. If $B = P^{-1}AP$ then

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P.$$

We compute

$$\begin{aligned}\det(B - \lambda I) &= \det(P^{-1}(A - \lambda I)P) \\ &= \det(P^{-1})\det(A - \lambda I)\det(P).\end{aligned}$$

Since $\det(P^{-1})\det(P) = \det(P^{-1}P) = \det(I) = 1$, we see that $\det(B - \lambda I) = \det(A - \lambda I)$. $\square$

Remark 14. *Note that matrices that have the same eigenvalues might not be similar: for instance, the matrices* $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ *and* $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ *have the same eigenvalues but are not similar.*

Remark 15. *Similarity is not the same as row equivalence. (If* A *is row equivalent to* B*, then* $B = EA$ *for some invertible matrix* E*.) Row operations on a matrix usually change its eigenvalues.*

We can use eigenvalues and eigenvectors to analyze the evolution of a dynamical system.

Example 16. *Let* $A = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}$. *Analyze the long-term behavior of the dynamical system defined by* $\mathbf{x}_{k+1} = A\mathbf{x}_k$ *(*$k = 0, 1, 2, \ldots$*) with* $\mathbf{x}_0 = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$.