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What is on today

1 Diagonalization

Lay–Lay–McDonald §5.3 pp. 283 – 288

Diagonal matrices make some computations much easier, as the following example illustrates:

Example 1. *Let* $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$. *What is* D^2*? What is* D^k*?*

$D^2 = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} 5^2 & 0 \\ 0 & 3^2 \end{pmatrix}$

$D^k = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}^k = \begin{pmatrix} 5^k & 0 \\ 0 & 3^k \end{pmatrix}$

What is D^{-1}?

$\frac{1}{15}\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$

$= \begin{pmatrix} 1/5 & 0 \\ 0 & 1/3 \end{pmatrix}$

$= \begin{pmatrix} 5^{-1} & 0 \\ 0 & 3^{-1} \end{pmatrix} \rightsquigarrow \underset{\shortparallel}{D^{-k}} (D^{-1})^k = \begin{pmatrix} 5^{-k} & 0 \\ 0 & 3^{-k} \end{pmatrix}$

$D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $D^{-1} = \frac{1}{\det D}\cdot\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

(recall: D^{-1} satisfies $D^{-1}\cdot D = I = D\cdot D^{-1}$)

If $A = PDP^{-1}$ for some invertible P and diagonal D, then A^k is also easy to compute.

Example 2. *Let* $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. *Find a formula for* A^k*, given that* $A = PDP^{-1}$*, where* $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

Find $A^2 = (PDP^{-1})(PDP^{-1}) = PD\cancel{P^{-1}P}DP^{-1} = P\cdot D^2\cdot P^{-1}$

$A^3 = (PDP^{-1})(PDP^{-1})(PDP^{-1}) = PD\cancel{P^{-1}P}D\cancel{P^{-1}P}DP^{-1} = PD^3P^{-1}$

(keep these – matrix multiplication does not commute)

$\vdots$

$A^k = PD^kP^{-1}$

So $\begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 5^k & 0 \\ 0 & 3^k \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 5^k & 3^k \\ -1\cdot 5^k & -2\cdot 3^k \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$

$= \boxed{\begin{pmatrix} 2\cdot 5^k - 1\cdot 3^k & 5^k - 1\cdot 3^k \\ -2\cdot 5^k + 2\cdot 3^k & -1\cdot 5^k + 2\cdot 3^k \end{pmatrix}}$

$P^{-1} = \frac{1}{\det P}\begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$

$= -1\cdot\begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$

e.g. A is diagonalizable in Ex. 2

A square matrix A is said to be *diagonalizable* if A is similar to a diagonal matrix: that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D. The next result gives us a characterization of diagonalizable matrices and how to construct a factorization.

Theorem 3 (Diagonalization). *An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.*

In other words, A is diagonalizable if and only if there are enough eigenvectors to form a basis of $\mathbb{R}^n$. We call such a basis an *eigenvector basis* of $\mathbb{R}^n$.

Example 4. *Diagonalize the matrix* $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$, *if possible.*

1 want to find: does A have 3 lin. indep. eigenvectors?

2 Need to find eigenvalues, then their corresponding eigenvectors ...

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = (1-\lambda)\left((-5-\lambda)(1-\lambda)+9\right) - -3\left(3(1-\lambda)-3\cdot 3\right) + 3\left(-9-3(-5-\lambda)\right)$$

$$= (1-\lambda)(-5-\lambda+5\lambda+\lambda^2+9) + 3(3-3\lambda-9) + 3(-9+15+3\lambda)$$

$$= (1-\lambda)(4+4\lambda+\lambda^2) + 3(-6-3\lambda) + 3(6+3\lambda)$$

$$= (1-\lambda)(\lambda+2)^2 + \cancel{-9(2+\lambda) + 9(2+\lambda)}$$

$\Rightarrow \lambda = 1$, or $\lambda = -2$.
(mult. 1) (mult. 2)

Now we need to find eigenvectors corresponding to

3a $\lambda = 1$ 3b $\lambda = -2$

$\lambda = 1$

$Av = \lambda v$

$(A-\lambda I)v = 0$

Find Null$(A-1I)$:

$$\begin{pmatrix} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_1 + x_2 = 0 \Rightarrow x_1 = -x_2 = x_3$

$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$

$$\begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$v_1 := \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$\lambda = -2$

Find Null$(A-2I)$:

$$\begin{pmatrix} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_1 = -x_2 - x_3$

$$\begin{pmatrix} -x_2-x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$v_2 := \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 := \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

4 Check that $\{v_1, v_2, v_3\}$ are linearly indep. (or: see Thm at end of class today) Thm 7(3)

5 $A = PDP^{-1}$, where $P = [v_1 \; v_2 \; v_3]$

must be invertible! (that's why we want the 3 eigenvectors to be lin indep.)

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

3 eigenvectors corresponding eigenvalues

Example 5. *Diagonalize the matrix* $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$, *if possible.*

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = (2-\lambda)\big((-6-\lambda)(1-\lambda)+9\big) + 4\big(4(1-\lambda)-9\big) + 3\big(-12-3(-6-\lambda)\big)$$

$$= (2-\lambda)(-6-\lambda+6\lambda+\lambda^2+9) + 4(4-4\lambda-9) + 3(-12+18+3\lambda)$$

$$= (2-\lambda)(3+5\lambda+\lambda^2) + 4(-5-4\lambda) + 3(6+3\lambda)$$

$$= 6-3\lambda+10\lambda-5\lambda^2+2\lambda^2-\lambda^3-20-16\lambda+18+9\lambda$$

$$= 4 + -3\lambda^2 - \lambda^3$$

$$= -1\cdot(\lambda^3+3\lambda^2-4)$$

$$= -1(\lambda-1)(\lambda^2+4\lambda+4) = -1(\lambda-1)(\lambda+2)^2$$

$$\Rightarrow \lambda = 1, \quad \lambda = -2$$

(mult. 1) (mult. 2)

$$\begin{array}{r|l} & \lambda^2+4\lambda+4 \\ \hline \lambda-1 & \lambda^3+3\lambda^2+0\lambda-4 \\ & -(\lambda^3-\lambda^2) \\ \hline & 4\lambda^2+0\lambda \\ & -(4\lambda^2-4\lambda) \\ \hline & 4\lambda-4 \\ & -(4\lambda-4) \\ \hline & 0 \end{array}$$

Basis for Null $(A-1\cdot I)$

$$\begin{pmatrix} 1 & 4 & 3 & 0 \\ -4 & -7 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 3 & 0 \\ 0 & 9 & 9 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 3 & 0 \\ 0 & 9 & 9 & 0 \\ 0 & 3 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow x_1 = -4x_2 - 3x_3 = -4(-x_3) - 3x_3 = 4x_3 - 3x_3 = x_3$$

$$x_2 = -x_3$$

$$\Rightarrow \begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Basis for Null $(A+2\cdot I)$

$$\begin{pmatrix} 4 & 4 & 3 & 0 \\ -4 & -4 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 4 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_3 = 0$$

$$x_1 = -x_2$$

$$\begin{pmatrix} -x_2 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

*note: multiplicity 2 for eigenvalue $\lambda = -2$, but eigenbasis was just one-dim'l.

Only have 2 eigenvectors, thus it's not diagonalizable!

The following theorem provides a sufficient condition for a matrix to be diagonalizable:

Theorem 6. *An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.*

However, it is not necessary for an $n \times n$ matrix to have n distinct eigenvalues in order to be diagonalizable!

Here is how we handle matrices whose eigenvalues are not distinct:

Theorem 7. *Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.*

1. *For $1 \le k \le p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.*
2. *The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens iff a) the characteristic polynomial factors completely into linear factors and b) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.*
3. *If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.*

Example 8. *Diagonalize the matrix* $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$, *if possible.*

Eigenvalues of triangular matrix: $\lambda = 5$ (mult. 2), $\lambda = -3$ (mult. 2)

Null $(A - 5I)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & -8 & 0 & 0 \\ -1 & -2 & 0 & -8 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & -8 & 0 & 0 \\ 0 & 2 & -8 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & -8 & 0 & 0 \\ 0 & 1 & -4 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_1 = -4x_2 + 8x_3 =$

$x_2 = 4x_3 + 4x_4$

$-4(4x_3 + 4x_4) + 8x_3$

$= -8x_3 - 16x_4$

$$\begin{pmatrix} -16x_4 - 8x_3 \\ 4x_3 + 4x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \underbrace{\begin{pmatrix} -8 \\ 4 \\ 1 \\ 0 \end{pmatrix}}_{= v_1} + x_4 \underbrace{\begin{pmatrix} -16 \\ 4 \\ 0 \\ 1 \end{pmatrix}}_{= v_2}$$

Null $(A + 3 \cdot I)$

$$\begin{pmatrix} 8 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow x_2 = 0$

$x_1 = 0$

x_3, x_4 free.

$$\underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{= v_3}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{= v_4}$$

By prev thm, the 4 vectors are lin. indep.

$$P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$A = PDP^{-1}$.