Today we explore length, distance, and perpendicularity for vectors in $\mathbb{R}^n$. All three ideas are defined in terms of the inner product of two vectors.

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then we can think of them as $n \times 1$ matrices. The transpose $\mathbf{u}^T$ is a $1 \times n$ matrix, and the matrix product $\mathbf{u}^T \mathbf{v}$ is a $1 \times 1$ matrix, which is a scalar. This scalar is called the inner product of $\mathbf{u}$ and $\mathbf{v}$ and is often written as $\mathbf{u} \cdot \mathbf{v}$ and called the dot product. If $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, then the inner product of $\mathbf{u}$ and $\mathbf{v}$ is

$$\begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + \cdots + u_nv_n.$$

**Example 1.** Let $\mathbf{u} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$. Compute $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$.

$$\mathbf{u} \cdot \mathbf{v} = (2)(3) + (-5)(2) + (-1)(-3) = 6 - 10 + 3 = -1$$

$$\mathbf{v} \cdot \mathbf{u} = (3)(2) + (2)(-5) + (-3)(-1) = 6 - 10 + 3 = -1$$

Here are properties of the inner product:

**Theorem 2.** Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$, and let $c$ be a scalar. Then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

4. $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$. 

Suppose $\mathbf{u} \cdot \mathbf{u} = 0$

$$\begin{cases} u_i = 0 & \text{for all } i \end{cases}$$

If $\mathbf{u} = (0, 0, \ldots, 0)$, then $\mathbf{u} \cdot \mathbf{u} = 0$. 

Suppose $\mathbf{u} \cdot \mathbf{u} = 0$.

$$\mathbf{u} = (u_1, u_2, \ldots, u_n)$$

Then $u_1^2 + u_2^2 + \cdots + u_n^2 = 0 \Rightarrow u_i = 0$ for all $i$. 


Definition 3. Let \( \mathbf{v} \) be a vector in \( \mathbb{R}^n \) with entries \( v_1, \ldots, v_n \). The length (or norm) of \( \mathbf{v} \) is the nonnegative scalar \( \| \mathbf{v} \| \) defined by
\[
\| \mathbf{v} \| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2},
\]
and \( \| \mathbf{v} \|^2 = \mathbf{v} \cdot \mathbf{v} \).

Note that if \( \mathbf{v} \in \mathbb{R}^2 \), then \( \| \mathbf{v} \| \) coincides with the standard notion of the length of the line segment from the origin to \( \mathbf{v} \) by the Pythagorean Theorem.

For any scalar \( c \), we have
\[
\| c \mathbf{v} \| = |c| \| \mathbf{v} \|.
\]

A vector whose length is 1 is called a unit vector. If we divide a nonzero vector \( \mathbf{v} \) by its length, we obtain a unit vector \( \mathbf{u} \) because the length of \( \mathbf{u} \) is \( \left( \frac{1}{\| \mathbf{v} \|} \right) \| \mathbf{v} \| \). The process of creating \( \mathbf{u} \) from \( \mathbf{v} \) is called normalizing \( \mathbf{v} \) and we say that \( \mathbf{u} \) is in the same direction as \( \mathbf{v} \).

Example 4. Let \( \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \). Find a unit vector \( \mathbf{u} \) in the same direction as \( \mathbf{v} \).
\[
\| \mathbf{v} \| = \sqrt{1+4+4} = 3
\]
\[
\mathbf{u} = \frac{\mathbf{v}}{\| \mathbf{v} \|} = \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 0 \end{bmatrix}
\]

Example 5. Let \( W \) be the subspace of \( \mathbb{R}^2 \) spanned by \( \mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \). Find a unit vector \( \mathbf{z} \) that is a basis for \( W \).
\[
\| \mathbf{z} \| = \sqrt{\mathbf{z} \cdot \mathbf{z}} = \sqrt{\left( \frac{2}{3} \right)^2 + 1} = \sqrt{\frac{13}{3}}; \quad \mathbf{z} = \frac{\mathbf{x}}{\| \mathbf{x} \|} = \frac{3}{\sqrt{13}} \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{13} \\ 3/\sqrt{13} \end{bmatrix}.
\]

Recall that if \( a, b \) are real numbers, the distance on the number line between \( a \) and \( b \) is given by the absolute value \( |a - b| \). This definition of distance in \( \mathbb{R} \) has a direct analogue in \( \mathbb{R}^n \).

Definition 6. For \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{R}^n \), the distance between \( \mathbf{u} \) and \( \mathbf{v} \), written as \( \text{dist}(\mathbf{u}, \mathbf{v}) \), is the length of the vector \( \mathbf{u} - \mathbf{v} \). That is, \( \text{dist}(\mathbf{u}, \mathbf{v}) = \| \mathbf{u} - \mathbf{v} \| \).

Example 7. Compute the distance between the vectors \( \mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \) and \( \mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \).
\[
\| \mathbf{u} - \mathbf{v} \| = \| \begin{bmatrix} 4 \\ -1 \end{bmatrix} \| = \sqrt{4^2 + (-1)^2} = \sqrt{17}.
\]

Definition 8. Two vectors \( \mathbf{u}, \mathbf{v} \in \mathbb{R}^n \) are orthogonal if \( \mathbf{u} \cdot \mathbf{v} = 0 \).

Note that the zero vector is orthogonal to every vector in \( \mathbb{R}^n \).

Here is a useful result about orthogonality:
Theorem 9 (Pythagorean Theorem). Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if
$$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2.$$  

If a vector $\mathbf{z}$ is orthogonal to every vector in a subspace $W$ of $\mathbb{R}^n$, then $\mathbf{z}$ is said to be orthogonal to $W$. The set of all vectors $\mathbf{z}$ that are orthogonal to $W$ is called the orthogonal complement of $W$ and is denoted by $W^\perp$.

Let $W$ be a subspace of $\mathbb{R}^n$. Here are two facts about orthogonal complements.

1. A vector $\mathbf{x}$ is in $W^\perp$ iff $\mathbf{x}$ is orthogonal to every vector in a set that spans $W$.
2. $W^\perp$ is a subspace of $\mathbb{R}^n$.

Here is another relationship between the null space and column space of a matrix.

Theorem 10. Let $A$ be an $m \times n$ matrix. The orthogonal complement of the column space of $A$ is the null space of $A^T$: $(\text{Col } A)^\perp = \text{Nul } A^T$.

2 Orthogonal sets

Lay–Lay–McDonald §6.2 pp. 340 – 342

A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal.

Example 11. Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set, where $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -1 \\ -2 \\ 7 \end{bmatrix}$.

Show:

\[
\begin{aligned}
\mathbf{u}_1 \cdot \mathbf{u}_2 &= 0 \\
\mathbf{u}_1 \cdot \mathbf{u}_3 &= 0 \\
\mathbf{u}_2 \cdot \mathbf{u}_3 &= 0 \\
\mathbf{u}_1 \cdot \mathbf{u}_2 &= 3(-1) + 1(2) + 1(1) = -3 + 2 + 1 = 0 \\
\mathbf{u}_1 \cdot \mathbf{u}_3 &= -3 + 2 + 7 = 0 \\
\mathbf{u}_2 \cdot \mathbf{u}_3 &= \frac{1}{2} - 4 + \frac{7}{2} = 0
\end{aligned}
\]

Theorem 12. If $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then $S$ is linearly independent and hence is a basis for the subspace spanned by $S$.

Proof. If $0 = c_1 \mathbf{u}_1 + \cdots + c_p \mathbf{u}_p$, then

\[
0 = 0 \cdot \mathbf{u}_1 = (c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \cdots + c_p \mathbf{u}_p) \cdot \mathbf{u}_1
\]

\[
= (c_1 \mathbf{u}_1) \cdot \mathbf{u}_1 + (c_2 \mathbf{u}_2) \cdot \mathbf{u}_1 + \cdots + (c_p \mathbf{u}_p) \cdot \mathbf{u}_1
\]

\[
= c_1 (\mathbf{u}_1 \cdot \mathbf{u}_1) + c_2 (\mathbf{u}_2 \cdot \mathbf{u}_1) + \cdots + c_p (\mathbf{u}_p \cdot \mathbf{u}_1)
\]

\[
= c_1 (\mathbf{u}_1 \cdot \mathbf{u}_1)
\]

because $\mathbf{u}_1$ is orthogonal to $\mathbf{u}_2, \ldots, \mathbf{u}_p$. Since $\mathbf{u}_1$ is nonzero, $\mathbf{u}_1 \cdot \mathbf{u}_1$ is not zero, and so $c_1 = 0$. Similarly, $c_2, \ldots, c_p$ must be zero. Thus $S$ is linearly independent.
Definition 13. An orthogonal basis for a subspace $W$ of $\mathbb{R}^n$ is a basis for $W$ that is also an orthogonal set.

The next theorem tells us why an orthogonal basis is nicer than other bases. The weights in a linear combination can be computed easily.

Theorem 14. Let $\{u_1, \ldots, u_p\}$ be an orthogonal basis for a subspace $W$ of $\mathbb{R}^n$. For each $y$ in $W$, the weights in the linear combination

$$y = c_1 u_1 + \cdots + c_p u_p$$

are given by

$$c_j = \frac{y \cdot u_j}{u_j \cdot u_j}, \quad j = 1, \ldots, p.$$  

Example 15. The set $\{u_1, u_2, u_3\}$ in the previous example is an orthogonal basis for $\mathbb{R}^3$.

Express the vector $y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as a linear combination of the vectors in $S$.

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ -8 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}, \quad y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$$

$$c_1 = \frac{y \cdot u_1}{u_1 \cdot u_1} = \frac{18 + 1 - 8}{9 + 4 + 64} = \frac{11}{64} = 1; \quad c_2 = \frac{y \cdot u_2}{u_2 \cdot u_2} = \frac{-6 + 2 - 8}{1 + 4 + 1} = -\frac{12}{6} = -2$$

$$c_3 = \frac{y \cdot u_3}{u_3 \cdot u_3} = \frac{-3 - 2 - 8}{4 + 4 + 16} = \frac{-33}{24} = -\frac{11}{8}$$

$$y = u_1 - 2u_2 - 2u_3$$

Given a nonzero vector $u$ in $\mathbb{R}^n$, we consider the problem of decomposing a vector $y \in \mathbb{R}^n$ into the sum of two vectors, one a multiple of $u$ and the other orthogonal to $u$. That is, we want to write

$$y = \hat{y} + z,$$

where $\hat{y} = \alpha u$ for some scalar $\alpha$ and $z$ is orthogonal to $u$. Equation (1) is satisfied under these constraints if and only if $\alpha = \frac{y \cdot u}{u \cdot u}$ and

$$\hat{y} = \frac{y \cdot u}{u \cdot u} u.$$  

The vector $\hat{y}$ is called the orthogonal projection of $y$ onto $u$ and the vector $z$ is called the component of $y$ orthogonal to $u$.

Example 16. Let $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$, $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection of $y$ onto $u$. Then write $y$ as the sum of two orthogonal vectors, one in $\text{Span}\{u\}$ and one orthogonal to $u$.  