Final review 04/27/21

lecture 17

Let $A = \begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix}$. Analyze the behavior of $A^k x_0$ for $k \gg 0$,

where $x_0 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$.

[Idea: Use eigenvalues & eigenvectors to compute $\lim_{k\to\infty} A^k x_0$.]

Step 1: Compute eigenvalues of A.

$$\det(A - \lambda I) = \begin{vmatrix} 0.95-\lambda & 0.03 \\ 0.05 & 0.97-\lambda \end{vmatrix} = (0.95-\lambda)(0.97-\lambda) - 0.03(0.05)$$

$$= \lambda^2 - 1.92\lambda + 0.92.$$ ← set this equal to 0 (characteristic polynomial)

Have $\lambda^2 - 1.92\lambda + 0.92 = (\lambda-1)(\lambda-0.92) \overset{=0}{\Rightarrow} \lambda = 1, 0.92.$

Step 2: For each eigenvalue, find corresponding eigenvectors.

Take $\lambda = 1$:

$$\begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix} v_1 = 1 \cdot v_1 \Rightarrow \begin{pmatrix} -0.05 & 0.03 \\ 0.05 & -0.03 \end{pmatrix} v_1 = 0$$

$\Rightarrow$ can take $v_1 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ ← this is an eigenvector corresponding to eigenvalue $\lambda = 1$.

(or any nonzero scalar multiple)

Take $\lambda = 0.92$

$$\begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix} v_2 = 0.92 v_2 \Rightarrow \begin{pmatrix} 0.03 & 0.03 \\ 0.05 & 0.05 \end{pmatrix} v_2 = 0$$

$\Rightarrow$ can take $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ← this is an eigenvector corresponding to eigenvalue $\lambda = 0.92$.

(or any nonzero scalar multiple)

Step 3: write x_0 in terms of eigenvector basis $\{v_1, v_2\}$.

$x_0 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$

$$\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = a\begin{pmatrix} 3 \\ 5 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 5 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

$$= \begin{pmatrix} 0.125 \\ 0.225 \end{pmatrix}$$

So $\begin{pmatrix}0.6\\0.4\end{pmatrix} = 0.125\begin{pmatrix}3\\5\end{pmatrix} + 0.225\begin{pmatrix}1\\-1\end{pmatrix}$

$x_0 = 0.125\, v_1 + 0.225\, v_2$.

Step 4 : Recall : $x_1 = Ax_0$
$x_2 = Ax_1$
$x_3 = Ax_2$
etc.

$$x_1 = A(x_0) = A(0.125v_1 + 0.225v_2)$$
$$= 0.125\,\underline{\underline{Av_1}} + 0.225\,Av_2$$
$$= 0.125\cdot 1 v_1 + 0.225\,(0.92)v_2$$
$$\vdots$$
$$x_k = A^k x_0 = 0.125\cdot 1^k v_1 + 0.225\,(0.92)^k v_2$$
$$x_k = 0.125\, v_1 + 0.225(0.92)^k v_2 .$$

As $k \gg 0$, $0.92^k \to 0 \Rightarrow x_k = 0.125 v_1$
$$= 0.125\begin{pmatrix}3\\5\end{pmatrix}.$$

Cramer's rule and inverses via cofactors

Lecture 10

Cramer's rule lets us solve systems of linear equations using determinants

If A is an invertible matrix:

$\boxed{A^{-1} = \frac{1}{\det A}\operatorname{adj} A}$, where adj A (adjugate of A) is constructed using cofactors of A.

Ex. Find inverse of $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$C_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix}$ $\quad C_{12} = -1\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$ $\quad C_{13} = +1\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix}$

$C_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix}$ $\quad C_{22} = +1\cdot\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}$ $\quad C_{23} = -1\cdot\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix}$

$C_{31} = 1\cdot\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}$ $\quad C_{32} = -1\cdot\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}$ $\quad C_{33} = +1\cdot\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$

$$\operatorname{adj} A = \begin{pmatrix} -2 & 3 & 5 \\ 14 & -7 & -7 \\ 4 & 1 & -3 \end{pmatrix}^T = \begin{pmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{pmatrix}$$

$$(\operatorname{adj} A)(A) = \begin{pmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{pmatrix} = \begin{pmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{pmatrix} = \underline{\underline{14\, I}} \Rightarrow \det A = 14.$$

this is a good sanity check – if you don't get $N \cdot I$ for some number N, something is wrong!

So we get $\boxed{A^{-1} = \frac{1}{14} \cdot \begin{pmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{pmatrix}}$