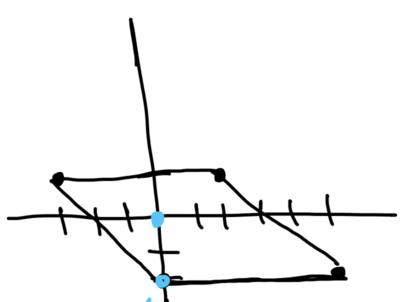
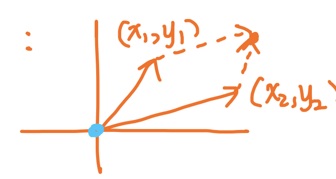
Final Exam Review 04/29/21

9) Find the area of the parallelogram with vertices $(0,-2)$, $(5,-2)$, $(-3,1)$, $(2,1)$.



move the vertex $(0,-2)$ to $(0,0)$; add $(0,+2)$

We saw :



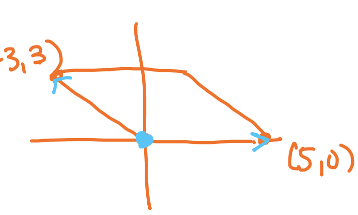
area of parallelogram :

$$\left\| \begin{matrix} x_1 & x_2 \\ y_1 & y_2 \end{matrix} \right\|$$

Do that for all vertices :

$(-3,1) + (0,2) \to (-3,3)$

$(5,-2) + (0,2) \to (5,0)$



$$\left| \begin{matrix} 5 & -3 \\ 0 & 3 \end{matrix} \right| = 15$$

So the area of the parallelogram is 15.

11) The first four Laguerre polynomials are 1, $1-t$, $2-4t+t^2$, $6-18t+9t^2-t^3$.
Show that these polynomials form a basis of P_3.

Translate all 4 polynomials into vectors :

$1 \to \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad 1-t \to \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$, $\quad 2-4t+t^2 \to \begin{pmatrix} \\ \end{pmatrix}$, $\quad 6-18t+9t^2-t^3 \to \begin{pmatrix} \\ \end{pmatrix}$

Now show that this set of 4 vectors is a basis for $\mathbb{R}^4$.

16) Describe how you might try to build a solution of a difference equation $x_{k+1} = Ax_k$ $(k=0,1,2,\dots)$ if you're given the initial x_0 and this vector did not happen to be an eigenvector of A.

If it's not an eigenvector, then rewrite in terms of eigenvectors. → need to compute eigenvalues & eigenvectors.

Say $\{v_1, \dots, v_n\}$ is basis of eigenvectors. Write $x_0 = \sum a_i v_i$.

Then take
$$\begin{aligned} x_1 &= Ax_0 \\ &= A\sum a_i v_i \\ &= \sum a_i Av_i \\ &= \sum a_i \lambda_i v_i \end{aligned}$$

Repeat:
$$\begin{aligned} x_2 &= Ax_1 \\ &= A(\sum a_i \lambda_i v_i) \\ &= \sum a_i \lambda_i Av_i \\ &= \sum a_i \lambda_i^2 v_i \end{aligned}$$

and so on.

20) Let $A = \begin{pmatrix} 0.4 & -0.3 \\ 0.4 & 1.2 \end{pmatrix}$. Explain why $A^k \to \begin{pmatrix} -0.5 & -0.75 \\ 1.0 & 1.50 \end{pmatrix}$ as $k \to \infty$.

Idea: compute eigenvalues and eigenvectors of A and use this to diagonalize A. Then $A = PDP^{-1} \Rightarrow A^k = PD^kP^{-1}$.

Compute $\begin{vmatrix} 0.4-\lambda & -0.3 \\ 0.4 & 1.2-\lambda \end{vmatrix} = 0$

$\lambda^2 - 1.6\lambda + 0.6 = 0 \Rightarrow \lambda = 1, \lambda = 0.6.$ these are your eigenvalues

Now compute corresponding eigenvectors:

For $\lambda_1 = 1$:

$$\begin{pmatrix} 0.4 & -0.3 \\ 0.4 & 1.2 \end{pmatrix} v_1 = 1 \cdot v_1$$

$$\Rightarrow \left(\begin{array}{cc|c} -0.6 & -0.3 & 0 \\ 0.4 & 0.2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

For $\lambda_2 = 0.6$:

$$\begin{pmatrix} -0.2 & -0.3 & 0 \\ 0.4 & 0.6 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

Then $A = PDP^{-1}$ (compute P, D, P^{-1}) $\quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0.6 \end{pmatrix}$

$A^k = PD^kP^{-1}$ (multiply some stuff, take $\lim\limits_{k \to \infty}$).

23) A Householder matrix has the form $Q = I - 2uu^T$ where u is a unit vector. Show that Q is an orthogonal matrix.

square ✓ invertible ✓ matrix s.t. Q^{-1} ✓ $= Q^T$ ✓

$$Q^TQ = (I - 2uu^T)^T (I - 2uu^T)$$

$$= \left(I^T - (2uu^T)^T\right)(I - 2uu^T)$$

$$= (I - 2uu^T)(I - 2uu^T) = I - 4uu^T + 4u\underbrace{u^Tu}_{=1}u^T$$

1 since u is a unit vector

$$= I - 4uu^T + 4uu^T = I.$$