The final exam will cover the following sections in the textbook:

§§1.1 – 1.5, 1.7 – 1.10, 2.1 – 2.3, 2.5, 3.1 – 3.3, 4.1 – 4.7, 5.1 – 5.3, 6.1 – 6.4.

Chapter 1. Linear Equations

- Solving systems of linear equations.
- Elementary row operations and (Reduced) Row Echelon Form.
- Pivot positions, pivot columns.
- Rewriting a linear system as a matrix equation $Ax = b$.
- Solutions of homogeneous equations $Ax = 0$.
- Solutions of the nonhomogeneous equation $Ax = b$ are obtained by taking a particular solution $x_0$ and adding all solutions of the homogeneous equation.
- Applications in business and science (§1.10).
- Linear independence of vectors.
- Linear transformations and their associated matrices.
- Linear transformations being one-to-one and onto; some properties are below.

Let $A$ be an $m \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^m$ the linear transformation given by $T(x) = Ax$.

<table>
<thead>
<tr>
<th>$T$ is one-to-one</th>
<th>$T$ is onto</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T(x) = b$ has <em>at most one solution</em> for every $b$. The columns of $A$ are linearly independent. $A$ has a pivot in every column.</td>
<td>$T(x) = b$ has <em>at least one solution</em> for every $b$. The columns of $A$ span $\mathbb{R}^m$. $A$ has a pivot in every row.</td>
</tr>
</tbody>
</table>

Chapter 2. Matrix Algebra

- Addition and multiplication of matrices.
- The inverse of a square matrix.
- $Ax = b$ has a unique solution if $A$ is invertible.
- $A \in M_{n \times n}$ is invertible iff its RREF is $I_n$. Know the algorithm for computing the inverse of a square matrix.
- The invertible matrix theorem: $A$ is invertible iff $A$ is one-to-one iff $A$ is onto. Be sure you know that $A$ is one-to-one iff the homogenous equation has only the trivial solution iff the columns of $A$ are linearly independent. Also, $A$ is onto iff $Ax = b$ has a solution for all $b$ iff $\text{Col}(A)$ is all of $\mathbb{R}^n$.
- $(AB)^{-1} = B^{-1}A^{-1}$ if $A$ and $B$ are invertible. $(AB)^T = B^T A^T$. 

1
• LU factorization.

Chapter 3. Determinants

• Computing the determinant of an \( n \times n \) matrix using cofactors and using elementary row operations.
• \( A \) is invertible iff \( \det(A) \neq 0 \).
• \( \det(A) \)(\( \det(B) \)) = \( \det(AB) \).
• Cramer’s Rule.
• For linear transformations \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \), we have area of \( T(S) \) equals \( |\det(A)| \) times the area of \( S \) for reasonable sets \( S \); there is a similar result for volumes for transformations on \( \mathbb{R}^3 \).

Chapter 4. Vector Spaces

• Definition and basic properties of vector spaces.
• Subspaces of vector spaces.
• The span of a set of vectors \( \text{Span}\{v_1, \ldots, v_k\} \) is always a subspace.
• The null space \( \text{Null}(A) \) of a transformation \( A \); it is the set of solutions of the homogeneous equation \( Ax = 0 \). For a general linear transformation \( A : V \rightarrow W \) of vector spaces, the null space is called the kernel of \( A \).
• The column space \( \text{Col}(A) \) of a matrix; it is the span of the columns of \( A \), and it equals the range of \( A \).
• Remember: if \( A \in M_{m \times n} \), then \( A \) determines a linear transformation \( A : \mathbb{R}^n \rightarrow \mathbb{R}^m \), and \( \text{Null}(A) \) is a subspace of \( \mathbb{R}^n \), while \( \text{Col}(A) \) is a subspace of \( \mathbb{R}^m \).
• The definition of a basis as a linearly independent set that spans the vector space.
• The pivot columns of \( A \) (not the pivot columns of an REF form of \( A \)) form a basis of \( \text{Col}(A) \).
• A basis of \( \text{Null}(A) \) is given by our usual method of finding the solution set of \( Ax = 0 \) in vector parametric form.
• Know some examples of vector spaces such as \( \mathbb{R}^n \), spaces of polynomials, spaces of functions.
• Coordinate systems: the \( \mathcal{B} \)-coordinates \( [x]_{\mathcal{B}} \) of \( x \) with respect to a basis \( \mathcal{B} = \{b_1, \ldots, b_n\} \) are given by \( [x]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}x \), where \( P_{\mathcal{B}} = [b_1 \cdots b_n] \). It is often easier to solve \( P_{\mathcal{B}}[x]_{\mathcal{B}} = x \).
• The dimension of a vector space equals the number of elements in a basis.

• If a subset $B$ of a vector space of dimension $n$ has $n$ elements and is linearly independent, then $B$ is a basis. A set of vectors $\{b_1, \ldots, b_n\}$ is a basis of $\mathbb{R}^n$ iff the matrix $[b_1 \cdots b_n]$ is invertible.

• For $A \in \mathcal{M}_{m \times n}$, $\dim \text{Nul}(A) + \dim \text{Col}(A) = n$.

• Given a basis $B = \{b_1, \ldots, b_n\}$ of a vector space $V$, the coordinate map $V \to \mathbb{R}^n$ given by $x \mapsto [x]_B$ is an isomorphism.

• For bases $B = \{b_1, \ldots, b_n\}, C = \{c_1, \ldots, c_n\}$ of $\mathbb{R}^n$, the relationship between the $B$ coordinates and the $C$ coordinates of a vector $x$ is given by

$$[x]_C = P_{C-B}[x]_B.$$ 

Chapter 5. Eigenvalues and Eigenvectors

• Definition of eigenvalues and eigenvectors.

• Eigenvectors belonging to distinct eigenvalues are linearly independent.

• Be able to use the characteristic equation to find eigenvalues.

• Diagonalization: $A = PDP^{-1}$ (this is possible if $A$ has $n$ distinct eigenvalues). Here the columns of $P$ are the eigenvectors, and the entries of the diagonal matrix $D$ are the eigenvalues. Remember: find the eigenvalues first from the characteristic equation, then find the eigenvectors.

• How to find $A^kx$ for $k \gg 0$ once you know a basis consisting of eigenvectors of $A$.

Chapter 6. Orthogonality

• Inner product on $\mathbb{R}^n$.

• Lengths of vectors; distance between vectors.

• Orthogonal vectors and orthogonal complements to subspaces.

• Orthogonal and orthonormal bases; properties of matrices with orthonormal columns.

• Orthogonal projections of vectors into subspaces.

• The Best Approximation Theorem: the best approximation to $y$ in a subspace $W$ is $\hat{y} = \text{proj}_W y$.

• Gram-Schmidt: constructing orthogonal and orthonormal bases.

• QR factorization.