The final exam will cover the following sections in the textbook:

$$\S\S 1.1 - 1.5, 1.7 - 1.10, 2.1 - 2.3, 2.5, 3.1 - 3.3, 4.1 - 4.7, 5.1 - 5.3, 6.1 - 6.4.$$

Chapter 1. Linear Equations

- Solving systems of linear equations.
- Elementary row operations and (Reduced) Row Echelon Form.
- Pivot positions, pivot columns.
- Rewriting a linear system as a matrix equation $A\mathbf{x} = \mathbf{b}$.
- Solutions of homogeneous equations $A\mathbf{x} = \mathbf{0}$.
- Solutions of the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ are obtained by taking a particular solution $\mathbf{x}_0$ and adding all solutions of the homogeneous equation.
- Applications in business and science (§1.10).
- Linear independence of vectors.
- Linear transformations and their associated matrices.
- Linear transformations being one-to-one and onto; some properties are below.

Let A be an $m \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^m$ the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$.

T is one-to-one	T is onto
$T(\mathbf{x}) = \mathbf{b}$ has *at most one solution* for every $\mathbf{b}$.	$T(\mathbf{x}) = \mathbf{b}$ has *at least one solution* for every $\mathbf{b}$.
The columns of A are linearly independent.	The columns of A span $\mathbb{R}^m$.
A has a pivot in every column.	A has a pivot in every row.

Chapter 2. Matrix Algebra

- Addition and multiplication of matrices.
- The inverse of a square matrix.
- $A\mathbf{x} = \mathbf{b}$ has a unique solution if A is invertible.
- $A \in M_{n\times n}$ is invertible iff its RREF is I_n. Know the algorithm for computing the inverse of a square matrix.
- The invertible matrix theorem: A is invertible iff A is one-to-one iff A is onto. Be sure you know that A is one-to-one iff the homogenous equation has only the trivial solution iff the columns of A are linearly independent. Also, A is onto iff $Ax = b$ has a solution for all b iff $\mathrm{Col}(A)$ is all of $\mathbb{R}^n$.
- $(AB)^{-1} = B^{-1}A^{-1}$ if A and B are invertible. $(AB)^T = B^TA^T$.

- LU factorization.

Chapter 3. Determinants

- Computing the determinant of an $n \times n$ matrix using cofactors and using elementary row operations.
- A is invertible iff $\det(A) \neq 0$.
- $(\det(A))(\det(B)) = \det(AB)$.
- Cramer's Rule.
- For linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$, we have area of $T(S)$ equals $|\det(A)|$ times the area of S for reasonable sets S; there is a similar result for volumes for transformations on $\mathbb{R}^3$.

Chapter 4. Vector Spaces

- Definition and basic properties of vector spaces.
- Subspaces of vector spaces.
- The span of a set of vectors $\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is always a subspace.
- The null space $\mathrm{Null}(A)$ of a transformation A; it is the set of solutions of the homogeneous equation $A\mathbf{x} = 0$. For a general linear transformation $A : V \to W$ of vector spaces, the null space is called the kernel of A.
- The column space $\mathrm{Col}(A)$ of a matrix; it is the span of the columns of A, and it equals the range of A.
- Remember: if $A \in \mathcal{M}_{m\times n}$, then A determines a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$, and $\mathrm{Null}(A)$ is a subspace of $\mathbb{R}^{\mathbf{n}}$, while $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^{\mathbf{m}}$.
- The definition of a basis as a linearly independent set that spans the vector space.
- The pivot columns of A (*not* the pivot columns of an REF form of A) form a basis of $\mathrm{Col}(A)$.
- A basis of $\mathrm{Null}(A)$ is given by our usual method of finding the solution set of $Ax = 0$ in vector parametric form.
- Know some examples of vector spaces such as $\mathbb{R}^n$, spaces of polynomials, spaces of functions.
- Coordinate systems: the $\mathcal{B}$-coordinates $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x}$ with respect to a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ are given by $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$, where $P_{\mathcal{B}} = [\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n]$. It is often easier to solve $P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$.

- The dimension of a vector space equals the number of elements in a basis.
- If a subset $\mathcal{B}$ of a vector space of dimension n has n elements and is linearly independent, then $\mathcal{B}$ is a basis. A set of vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis of $\mathbb{R}^n$ iff the matrix $[\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n]$ is invertible.
- For $A \in \mathcal{M}_{m \times n}$, dim Nul(A) + dim Col(A) = n.
- Given a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of a vector space V, the coordinate map $V \to \mathbb{R}^n$ given by $\mathbf{x} \mapsto [\mathbf{x}]_\mathcal{B}$ is an isomorphism.
- For bases $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ of $\mathbb{R}^n$, the relationship between the $\mathcal{B}$ coordinates and the $\mathcal{C}$ coordinates of a vector $\mathbf{x}$ is given by

$$[\mathbf{x}]_\mathcal{C} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\mathbf{x}]_\mathcal{B}.$$

Chapter 5. Eigenvalues and Eigenvectors

- Definition of eigenvalues and eigenvectors.
- Eigenvectors belonging to distinct eigenvalues are linearly independent.
- Be able to use the characteristic equation to find eigenvalues.
- Diagonalization: $A = PDP^{-1}$ (this is possible if A has n distinct eigenvalues). Here the columns of P are the eigenvectors, and the entries of the diagonal matrix D are the eigenvalues. Remember: find the eigenvalues first from the characteristic equation, then find the eigenvectors.
- How to find $A^k\mathbf{x}$ for $k \gg 0$ once you know a basis consisting of eigenvectors of A.

Chapter 6. Orthogonality

- Inner product on $\mathbb{R}^n$.
- Lengths of vectors; distance between vectors.
- Orthogonal vectors and orthogonal complements to subspaces.
- Orthogonal and orthonormal bases; properties of matrices with orthonormal columns.
- Orthogonal projections of vectors into subspaces.
- The Best Approximation Theorem: the best approximation to $\mathbf{y}$ in a subspace W is $\hat{\mathbf{y}} = \text{proj}_W \mathbf{y}$.
- Gram-Schmidt: constructing orthogonal and orthonormal bases.
- QR factorization.