Midterm 2 will cover the following sections in the textbook: §§$4.1 - 4.7, 5.1 - 5.3, 6.1 - 6.3$. Most questions will be computational. Some problems may ask you whether a certain statement is true or false and ask you to justify your answer.

Chapter 4. Vector Spaces

- Definition and basic properties of vector spaces.
- Subspaces of vector spaces.
- The span of a set of vectors, $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is always a subspace.
- The null space $\text{Null}(A)$ of a transformation A; it is the set of solutions of the homogeneous equation $A\mathbf{x} = 0$. For a general linear transformation $A : V \to W$ of vector spaces, the null space is called the kernel of A.
- The column space $\text{Col}(A)$ of a matrix; it is the span of the columns of A, and it equals the range of A.
- Remember: if $A \in \mathcal{M}_{m\times n}$, then A determines a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$, and $\text{Null}(A)$ is a subspace of $\mathbb{R}^{\mathbf{n}}$, while $\text{Col}(A)$ is a subspace of $\mathbb{R}^{\mathbf{m}}$.
- The definition of a basis as a linearly independent set that spans the vector space.
- The pivot columns of A (*not* the pivot columns of an REF form of A) form a basis of $\text{Col}(A)$.
- A basis of $\text{Null}(A)$ is given by our usual method of finding the solution set of $Ax = 0$ in vector parametric form.
- Know some examples of vector spaces such as $\mathbb{R}^n$, spaces of polynomials, spaces of functions.
- Coordinate systems: the $\mathcal{B}$-coordinates $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x}$ with respect to a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ are given by $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$, where $P_{\mathcal{B}} = [\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n]$. It is often easier to solve $P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$.
- The dimension of a vector space equals the number of elements in a basis.
- If a subset $\mathcal{B}$ of a vector space of dimension n has n elements and is linearly independent, then $\mathcal{B}$ is a basis. A set of vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis of $\mathbb{R}^n$ iff the matrix $[\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n]$ is invertible.
- For $A \in \mathcal{M}_{m\times n}$, dim Nul(A) + dim Col(A) = n.
- Given a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of a vector space V, the coordinate map $V \to \mathbb{R}^n$ given by $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is an isomorphism.

- For bases $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ of $\mathbb{R}^n$, the relationship between the $\mathcal{B}$ coordinates and the $\mathcal{C}$ coordinates of a vector $\mathbf{x}$ is given by

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\mathbf{x}]_{\mathcal{B}}.$$

Chapter 5. Eigenvalues and Eigenvectors

- Definition of eigenvalues and eigenvectors.
- Eigenvectors belonging to distinct eigenvalues are linearly independent.
- Be able to use the characteristic equation to find eigenvalues.
- Diagonalization: $A = PDP^{-1}$ (this is possible if A has n distinct eigenvalues). Here the columns of P are the eigenvectors, and the entries of the diagonal matrix D are the eigenvalues. Remember: find the eigenvalues first from the characteristic equation, then find the eigenvectors.
- How to find $A^k\mathbf{x}$ for $k \gg 0$ once you know a basis consisting of eigenvectors of A.

Chapter 6. Orthogonality

- Inner product on $\mathbb{R}^n$.
- Lengths of vectors; distance between vectors.
- Orthogonal vectors and orthogonal complements to subspaces.
- Orthogonal and orthonormal bases; properties of matrices with orthonormal columns.
- Orthogonal projections of vectors into subspaces.
- The Best Approximation Theorem: the best approximation to $\mathbf{y}$ in a subspace W is $\hat{\mathbf{y}} = \text{proj}_W \mathbf{y}$.