Midterm 2 will cover the following sections in the textbook: §§4.1 − 4.7, 5.1 − 5.3, 6.1 − 6.3. Most questions will be computational. Some problems may ask you whether a certain statement is true or false and ask you to justify your answer.

Chapter 4. Vector Spaces

- Definition and basic properties of vector spaces.
- Subspaces of vector spaces.
- The span of a set of vectors, \( \operatorname{Span}\{v_1, \ldots, v_k\} \) is always a subspace.
- The null space \( \operatorname{Null}(A) \) of a transformation \( A \); it is the set of solutions of the homogeneous equation \( Ax = 0 \). For a general linear transformation \( A : V \to W \) of vector spaces, the null space is called the kernel of \( A \).
- The column space \( \operatorname{Col}(A) \) of a matrix; it is the span of the columns of \( A \), and it equals the range of \( A \).
- Remember: if \( A \in \mathcal{M}_{m \times n} \), then \( A \) determines a linear transformation \( A : \mathbb{R}^n \to \mathbb{R}^m \), and \( \operatorname{Null}(A) \) is a subspace of \( \mathbb{R}^n \), while \( \operatorname{Col}(A) \) is a subspace of \( \mathbb{R}^m \).
- The definition of a basis as a linearly independent set that spans the vector space.
- The pivot columns of \( A \) (not the pivot columns of an REF form of \( A \)) form a basis of \( \operatorname{Col}(A) \).
- A basis of \( \operatorname{Null}(A) \) is given by our usual method of finding the solution set of \( Ax = 0 \) in vector parametric form.
- Know some examples of vector spaces such as \( \mathbb{R}^n \), spaces of polynomials, spaces of functions.
- Coordinate systems: the \( B \)-coordinates \( [x]_B \) of \( x \) with respect to a basis \( B = \{b_1, \ldots, b_n\} \) are given by \( [x]_B = P_B^{-1}x \), where \( P_B = [b_1 \quad \cdots \quad b_n] \). It is often easier to solve \( P_B[x]_B = x \).
- The dimension of a vector space equals the number of elements in a basis.
- If a subset \( B \) of a vector space of dimension \( n \) has \( n \) elements and is linearly independent, then \( B \) is a basis. A set of vectors \( \{b_1, \ldots, b_n\} \) is a basis of \( \mathbb{R}^n \) iff the matrix \( [b_1 \quad \cdots \quad b_n] \) is invertible.
- For \( A \in \mathcal{M}_{m \times n} \), \( \dim \operatorname{Nul}(A) + \dim \operatorname{Col}(A) = n \).
- Given a basis \( B = \{b_1, \ldots, b_n\} \) of a vector space \( V \), the coordinate map \( V \to \mathbb{R}^n \) given by \( x \mapsto [x]_B \) is an isomorphism.
• For bases $\mathcal{B} = \{b_1, \ldots, b_n\}$, $\mathcal{C} = \{c_1, \ldots, c_n\}$ of $\mathbb{R}^n$, the relationship between the $\mathcal{B}$ coordinates and the $\mathcal{C}$ coordinates of a vector $x$ is given by

$$ [x]_\mathcal{C} = P_{\mathcal{C} \rightarrow \mathcal{B}} [x]_\mathcal{B}. $$

Chapter 5. Eigenvalues and Eigenvectors

• Definition of eigenvalues and eigenvectors.

• Eigenvectors belonging to distinct eigenvalues are linearly independent.

• Be able to use the characteristic equation to find eigenvalues.

• Diagonalization: $A = PDP^{-1}$ (this is possible if $A$ has $n$ distinct eigenvalues). Here the columns of $P$ are the eigenvectors, and the entries of the diagonal matrix $D$ are the eigenvalues. Remember: find the eigenvalues first from the characteristic equation, then find the eigenvectors.

• How to find $A^kx$ for $k \gg 0$ once you know a basis consisting of eigenvectors of $A$.

Chapter 6. Orthogonality

• Inner product on $\mathbb{R}^n$.

• Lengths of vectors; distance between vectors.

• Orthogonal vectors and orthogonal complements to subspaces.

• Orthogonal and orthonormal bases; properties of matrices with orthonormal columns.

• Orthogonal projections of vectors into subspaces.

• The Best Approximation Theorem: the best approximation to $y$ in a subspace $W$ is $\hat{y} = \text{proj}_W y$. 