MA 841 (Greatest hits in arithmetic geometry)
Fall 2021
Instructor: Jennifer Balakrishnan
Time: TR 9:30 AM – 10:45 AM, SOC B61
Office hours: TR 12:00 PM – 1:30 PM, MCS 271

Course overview

This course will survey a selection of celebrated papers in arithmetic geometry. Many of these articles are well known because they posed questions that resulted in foundational lines of investigation, produced a spectacular technical breakthrough, or gave new perspective on a well-studied problem. But reading a paper for the first time is not always linear. How do we read papers in mathematics? With the goal of developing this skill, we will prepare expository talks and articles based on some of the most exciting developments in our field.

As a group, we will first select a subset of papers from the list on the following pages. (If there are further suggestions, please let me know!)

For each paper, we will assign the following roles[1]:

Mathematician,
Experimentalist, and
Historian/Science Journalist,

with the "Mathematician" role split between a pair of students and the "Experimentalist" role split between two students, depending on the style and length of the paper considered.

For each paper, I will start by giving an overview with background and context. The Mathematician will present the key ideas in the paper, and the Experimentalist will report on related computations. The role of the Historian/Science Journalist will be to give (aspirationally) a *Quanta Magazine*-level write-up of the paper based on the presentations, possibly supplemented by further reading. Everyone is expected to read the assigned materials before each class and actively participate.

NB: Optimistically, we will cover one paper approximately every one to two weeks in this style, but, of course, this depends on the papers we choose. Many of these papers deserve to be the topic of entire standalone semester-long seminars on their own! Our goal is more modest: to understand how to take apart a paper quickly, make black boxes, figure out how the main ideas hold together, and as necessary, describe the tools needed to open the black boxes.

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[1]With all credit for this framework due to Nick Trefethen, who famously led a numerical analysis course at Cornell in this manner: https://people.maths.ox.ac.uk/trefethen/classic_papers.txt

As for how the course will go, I will meet with the Mathematician(s) and Experimentalist(s) before their presentation to help prepare and suggest experiments. During the classes each week, the lectures will be split among Mathematician(s), Experimentalist(s), and myself, with a partition appropriate for the level of the paper.

By the end of the last lecture on the paper, the Historian/Science Journalist will draft a short article based on the lectures, share it with all of us, and we will collectively offer feedback on the article. To facilitate this, I will maintain a Dropbox folder of all articles and other associated readings and shared Overleaf and CoCalc projects for the write-ups and experiments.

Evaluation

Each student will actively participate in the course. The course grade will be based on the presentations (40%), written expository articles (40%), and participation (20%).

Potential papers

1. M. Bhargava and A. Shankar, "Binary quartic forms having bounded invariants, and the boundedness of the average rank of elliptic curves," *Ann. of Math. (2)* 181 (2015), no. 1, 191–242.
2. B. Birch and P. Swinnerton-Dyer, "Notes on Elliptic Curves (II)," *J. Reine Angew. Math*. 218 (1965), 79–108.
3. R. F. Coleman. "Effective Chabauty." *Duke Math. J*. 52 (3) 765 - 770, September 1985.
4. P. Deligne, "La conjecture de Weil. I.," *Inst. Hautes Études Sci. Publ. Math.* No. 43 (1974), 273–307.
5. P. Deligne and G. Lusztig, "Representations of reductive groups over finite fields," *Ann. of Math.* (2) 103 (1976), no. 1, 103–161.
6. N. Elkies, "The existence of infinitely many supersingular primes for every elliptic curve over **Q**," *Invent. Math.* 89 (1987) no. 3, 561–567.
7. M. Emerton, *p*-adic families of modular forms (after Hida, Coleman, and Mazur). *Séminaire Bourbaki*. Vol. 2009/2010. Exposés 1012–1026.
8. G. Faltings[2], "Endlichkeitssätze für abelsche Varietäten über Zahlkörpern," [Finiteness theorems for abelian varieties over number fields] *Invent. Math*. 73 (1983), no. 3, 349–366.
9. D. R. Hayes, "Explicit class field theory for rational function fields," *Trans. Amer. Math. Soc.* 189 (1974), 77–91.
10. T. Honda, "Isogeny classes of abelian varieties over finite fields," *J. Math. Soc. Japan*, vol number 1-2 (1968): 83-95, with J. Tate, "Endomorphisms of abelian varieties over finite fields," *Invent. Math.,* **2** (1966): 134-144. and J. Tate, "Classes d'isogénie des variétés abéliennes sur un corps fini," *Séminaire N. Bourbaki*, 1971, exp. no 352, 95-110.

[2] We will likely instead use parts of *Rational points.* Third edition. Papers from the seminar held at the Max-Planck-Institut für Mathematik, Bonn, 1983/1984. Edited by Faltings and Wüstholz. Aspects of Mathematics, E6. *Friedr. Vieweg & Sohn, Braunschweig; distributed by Heyden & Son, Inc., Philadelphia, PA,* 1992. x+311.

11. S. Kamienny and B. Mazur (with an appendix by A. Granville), "Rational torsion of prime order in elliptic curves over number fields, " *Astérisque*. 228: 81–100 (1995).
12. B. Mazur (with an appendix by D. Goldfeld), "Rational isogenies of prime degree," *Invent. Math.* 44 (1978), no. 2, 129–162.
13. B. Mazur, J. Tate, and J. Teitelbaum. On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer. *Invent Math* **84,** 1–48 (1986).
14. L. Merel, "Bornes pour la torsion des courbes elliptiques sur les corps de nombres," [Bounds for the torsion of elliptic curves over number fields]. *Invent. Math*. 124: (1996), no. 1, 437–449.
15. B. Poonen and M. Stoll, The Cassels-Tate pairing on polarized abelian varieties. *Ann. of Math.* (2) 150 (1999), no. 3, 1109–1149.
16. K. Ribet, "On Modular representations of Gal($\overline{\mathbf{Q}}/\mathbf{Q}$) arising from modular forms," *Invent. Math*. 100 (1990), no. 2, 431–476.
17. J.-P. Serre[3], "Sur les représentations modulaires de degré 2 de Gal($\overline{\mathbf{Q}}/\mathbf{Q}$)," [On modular representations of degree 2 of Gal($\overline{\mathbf{Q}}/\mathbf{Q}$)], *Duke Math. J*. 54 (1987), 179–230.
18. J. Tate, "Fourier analysis in number fields and Hecke's zeta functions," Ph.D. thesis, Princeton, 1950.

Schedule and deadlines

September 2, 2021: **No class** (Balakrishnan in Germany for a conference). Please look over the papers above and make a list of your top 8-10 papers that you'd like to see this semester (roughly ranked). *Note: If there is a paper you'd like to see that's not on the list, please feel free to nominate it during the discussion in class on September 7th.*

September 7, 2021: Introduction to the course, organizational discussion, choosing papers and possibly revising the schedule below, assigning roles for each paper. *Update Sept. 7: we chose the following eight papers: Elkies, Emerton, Mazur—Tate—Teitelbaum, Poonen—Stoll, Serre, Ribet, Hayes, and Deligne.*

September 9, 2021: Paper 1 (Elkies) presentation
September 14, 2021: Paper 1 presentation

September 16, 2021: Paper 2 (Emerton) presentation
September 21, 2021: Paper 2 presentation

September 23, 2021: Paper 3 (Mazur—Tate—Teitelbaum) presentation
September 28, 2021: Paper 3 presentation
September 30, 2021: Paper 3 presentation

October 5, 2021: Paper 4 (Poonen—Stoll) presentation
October 7, 2021: Paper 4 presentation

[3] English translation available courtesy of Alex Ghitza: https://aghitza.org/publications/translation_serre_duke

October 14, 2021: Paper 5 (Serre) presentation
October 19, 2021: Paper 5 presentation
October 21, 2021: Paper 5 presentation

October 26, 2021: Paper 6 (Ribet) presentation
October 28, 2021: Paper 6 presentation
November 2, 2021: Paper 6 presentation
November 4, 2021: Paper 6 presentation

November 9, 2021: Paper 7 (Hayes) presentation
November 11, 2021: Paper 7 presentation

November 16, 2021: Paper 8 (Deligne) presentation
November 18, 2021: Paper 8 presentation
November 23, 2021: Paper 8 presentation
November 30, 2021: Paper 8 presentation
December 2, 2021: Paper 8 presentation

December 7, 2021: Paper-writing wrap up, live editing
December 9, 2021: Course wrap up