MA 841 (Greatest hits in arithmetic geometry)
Fall 2021
Instructor: Jennifer Balakrishnan

Papers, Supplemental Readings, and Experiments

NB: The eight papers were chosen by the class, supplemental readings were suggested by all, and experiments were set by the Experimentalist(s) for each paper.

Paper 1

N. Elkies, "The existence of infinitely many supersingular primes for every elliptic curve over **Q**," *Invent. Math.* **89** (1987) no. 3, 561–567.

Supplemental reading:

- N. Elkies, "Supersingular primes for elliptic curves over real number fields," *Compositio Math.* **72** (1989), no. 2, 165–172.
- N. Elkies, "Distribution of supersingular primes," Journées Arithmétiques, 1989 (Luminy, 1989). *Astérisque* No. 198-200 (1991), 127–132 (1992).
- B. Poonen, "Drinfeld modules with no supersingular primes," *IMRN*, 1998, no. 3, 151–159.

Experiments:

- Replicating the example with X_0(11) and X_1(11) in Section 4 of the paper
- Computing Hilbert class polynomials following the paper of A. Sutherland, "Computing Hilbert class polynomials with the Chinese Remainder Theorem," *Math. Comp.* **80** (2011), 501-538.

Paper 2

M. Emerton, *p*-adic families of modular forms (after Hida, Coleman, and Mazur). *Séminaire Bourbaki*. Vol. 2009/2010. Exposés 1012–1026.

Supplemental reading:

- F. Calegari, "Congruences between modular forms," Lectures from the 2013 Arizona Winter School.
- J. Vonk, "Overconvergent modular forms and their explicit arithmetic," *Bull. Amer. Math. Soc.* (N.S.) **58** (2021), no. 3, 313–356.

Experiments:

- Verify Kummer congruences, apply them to construct *p*-adic zeta functions and *p*-adic modular forms

Paper 3

B. Mazur, J. Tate, and J. Teitelbaum. On *p*-adic analogues of the conjectures of Birch and Swinnerton-Dyer. *Invent Math* **84,** 1–48 (1986).

Supplemental reading:

- R. Greenberg, G. Stevens, "*p*-adic *L*-functions and *p*-adic periods of modular forms," *Invent. Math.* **111** (1993), no. 2, 407–447.
- R. Pollack, "Overconvergent modular symbols," *Computations with modular forms,* 69-105, Contrib. Math. Comput. Sci., 6, Springer, Cham, 2014.
- W. Stein, C. Wuthrich, "Algorithms for the arithmetic of elliptic curves using Iwasawa Theory," *Math. Comp.* **82** (2013), no. 283, 1757-1792.

Experiments:

- Computing *p*-adic *L*-series attached to elliptic curves over **Q**
- Computing *p*-primary parts of the Shafarevich-Tate group attached to elliptic curves over **Q**

Paper 4

B. Poonen and M. Stoll, The Cassels-Tate pairing on polarized abelian varieties. *Ann. of Math.* (2) **150** (1999), no. 3, 1109–1149.

Supplemental reading:

- M. van Beek, *Computing the Cassels-Tate pairing*, Cambridge PhD thesis, 2015.
- B. Conrad, "Polarizations," lectures notes from https://math.stanford.edu/~conrad/vigregroup/vigre04.html
- W. Stein, "Shafarevich-Tate groups of nonsquare order," *Modular curves and abelian varieties*, 277-289, *Progr. Math.,* 224, Birkhäuser, Basel, 2004.

Experiments:

- Use the Cassels-Tate pairing for elliptic curves to refine Mordell-Weil rank bounds, as in Magma: http://magma.maths.usyd.edu.au/magma/handbook/text/1514#17533
- Use Stein's Magma code (in Section 2 of his paper above) to verify Conjecture 2.1

Paper 5

J.-P. Serre[1], "Sur les représentations modulaires de degré 2 de Gal($\overline{\mathbf{Q}}/\mathbf{Q}$)," [On modular representations of degree 2 of Gal($\overline{\mathbf{Q}}/\ \mathbf{Q}$)], *Duke Math. J.* **54** (1987), 179–230.

Supplemental reading:

- A. J. Best, "Serre's conjecture," Cambridge Part III Essay, 2015.
- H. Darmon, "Serre's conjectures," *Seminar on Fermat's Last Theorem*, V. Kumar Murty ed., CMS conference proceedings vol 17., pp 135-155, 2007.
- K. A. Ribet, W. A. Stein, "Lectures on Serre's conjectures," *Arithmetic algebraic geometry (Park City, UT, 1999)*, 143-232, *IAS/Park City Math. Ser.*, **9**, AMS, Providence, RI, 2001.

Experiments:

- Find the modular form attached to the Galois representation rho: G -> GL_2 (F_5) arising from the dihedral extension of **Q** given by x^4 – 3 (see Example 5.2.1 of Best's Part III Essay).
- Mod 3 Galois representations arising from 3-torsion of elliptic curves

Paper 6

K. A. Ribet, "On Modular representations of Gal($\overline{\mathbf{Q}}/\mathbf{Q}$) arising from modular forms," *Invent. Math*. 100 (1990), no. 2, 431–476.

Supplemental reading:

- K. A. Ribet, "From the Taniyama-Shimura conjecture to Fermat's last theorem," *Ann. Fac. Sci. Toulouse Math.* (5) **11** (1990), no. 1, 116-139.

Experiments:

- Understanding the special fibres of the modular curve X_0(14) and the Shimura curve associated with an Eichler order of level 1 in the quaternion algebra over **Q** with discriminant 14
- Level-lowering Galois representations attached to elliptic curves and finding modular representations attached to elliptic curves
- Level-lowering Galois representations attached to elliptic curves and finding modular representations attached to higher-dimensional abelian varieties

[1] English translation available courtesy of Alex Ghitza: https://aghitza.org/publications/translation_serre_duke

Paper 7

D. R. Hayes, "Explicit class field theory for rational function fields," *Trans. Amer. Math. Soc.* 189 (1974), 77–91.

Supplemental reading:

- D. R. Hayes, "A brief introduction to Drinfel'd modules," *The arithmetic of function fields (Columbus, OH, 1991),* 1-32, OSU Math. Res. Inst. Publ., 2, de Gruyter, Berlin, 1992.

Experiments:

- Computing with Carlitz modules, as in https://magma.maths.usyd.edu.au/magma/handbook/text/493

Paper 8

P. Deligne, "La conjecture de Weil. I.," *Inst. Hautes Études Sci. Publ. Math.* No. 43 (1974), 273–307.

Supplemental reading:

- S. Chan, "Topics in the theory of zeta functions of curves," Oxford MMath dissertation, 2016.
- N. Katz, "An overview of Deligne's proof of the Riemann Hypothesis for varieties over finite fields," Mathematical developments arising from Hilbert problems (Proc. Sympos. Pure Math., Vol. XXVIII, Northern Illinois Univ., De Kalb, Ill., 1974), pp. 275-305. AMS, Providence, RI, 1976.
- K. S. Kedlaya, "Counting points on hyperelliptic curves using Monsky-Washnitzer cohomology," *J. Ramanujan Math. Soc.* **16** (2001), no. 4, 323-338.
- J. S. Milne, *Lectures on Étale Cohomology*, 2013, https://www.jmilne.org/math/CourseNotes/LEC.pdf

Experiments:

- Computing zeta functions of elliptic curves over finite fields: by hand and using Kedlaya's algorithm (following Section 4.7 of Chan's MMath dissertation)
- Computing zeta functions of hyperelliptic and Artin-Schreier curves over finite fields